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Paul A. Farrell
Kent State University - Kent Campus, farrell@cs.kent.edu

Eugene O'Riordan
Dublin City University

Grigori I. Shishkin
Russian Academy of Sciences

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A CLASS OF SINGULARLY PERTURBED QUASILINEAR DIFFERENTIAL EQUATIONS WITH INTERIOR LAYERS

P. A. FARRELL, E. O'RIORDAN, AND G. I. SHISHKIN

ABSTRACT. A class of singularly perturbed quasilinear differential equations with discontinuous data is examined. In general, interior layers will appear in the solutions of problems from this class. A numerical method is constructed for this problem class, which involves an appropriate piecewise-uniform mesh. The method is shown to be a parameter-uniform numerical method with respect to the singular perturbation parameter. Numerical results are presented, which support the theoretical results.

1. INTRODUCTION

Convection-diffusion equations of the form $(-\varepsilon u_x)_x + (b(u))_x = f(x)$, with a nonlinearity of the type $b(u) = u^2$, arise in numerous applications involving fluid dynamics. The Navier-Stokes equations involve such a nonlinearity, as do the drift-diffusion equations for modelling semiconductor devices. Depending on the specified boundary conditions, boundary and/or interior layers can arise in the solutions of such nonlinear equations. In this paper, we examine a class of nonlinear singularly perturbed ordinary differential equations, whose solutions exhibit interior layers. Moving interior layers are often associated with shock waves in gas dynamics. Burgers' equation is typically used as an initial mathematical model to study such shock layer phenomena. The nonlinear problem analysed in this paper can be viewed as a step towards understanding such classical nonlinearities.

In the case of a linear singularly perturbed ordinary differential equation, classical numerical methods usually give unsatisfactory numerical results, when the singular perturbation parameter ε is small. A parameter-uniform numerical method [4] is a numerical method for a singularly perturbed problem having an asymptotic error bound in the pointwise maximum norm that is independent of the size of the singular perturbation parameter. Parameter-uniform behaviour may be achieved by fitting the mesh [4] or by fitting the finite difference operator to the boundary/interior layer [13]. In [5] it was proved that for a class of singularly perturbed semilinear two point boundary value problems, parameter-uniform convergence in the maximum norm is not achievable using fitted operator schemes with frozen coefficients on uniform meshes. However, a parameter-uniform numerical method

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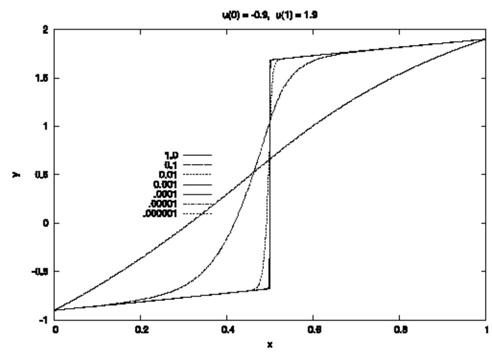

FIGURE 1. A representative set of solutions for different values of ε for a problem from the class of problems given in (1.1)

for such semilinear problems, which are of reaction-diffusion type, that combined a standard finite difference operator with a fitted piecewise-uniform mesh was given in [6]. In this paper, we construct a parameter-uniform method, based on a standard upwind finite difference operator and a fitted piecewise-uniform mesh, for a nonlinear convection-diffusion problem.

Farrell et al. [7], Linß et al. [10, 9], and Vulanović [15] examined quasilinear convection-diffusion problems where the problem class was such that only boundary layers occurred in the solutions. In the papers [7, 10, 15], parameter-uniform numerical methods were developed, which were based on Shishkin-type piecewise uniform meshes [14, 12, 4] for the problem. In this paper, we examine the case of a quasilinear convection-diffusion problem, where interior layers can occur in the solutions. An analytical discussion of quasilinear problems with interior layers is given in [2].

In this paper the following class of singularly perturbed quasilinear ordinary differential equations with discontinuous data is considered. Let $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$ and $\bar{\Omega} = [0,1]$. Find $u_\varepsilon \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$\varepsilon u_\varepsilon''(x) + b(x,u)u_\varepsilon'(x) = f(x),\ x \in \Omega^- \cup \Omega^+,\ u_\varepsilon(0) = A,\ u_\varepsilon(1) = B, \tag{1.1a}$$

$$b(x,u) = \begin{cases} b_1(u) = -1 + cu,\ x < d, \\ b_2(u) = 1 + cu,\ x > d, \end{cases} \qquad f(x) = \begin{cases} -\delta_1,\ x < d, \\ \delta_2,\ x > d, \end{cases} \tag{1.1b}$$

$$-1 < u_\varepsilon(0) < 0, \quad 0 < u_\varepsilon(1) < 1, \quad 0 < c \le 1, \tag{1.1c}$$

where δ_1, δ_2 are nonnegative constants. Note the strict inequalities in (1.1c). In order to study and analyse monotonically increasing solutions, we impose further conditions on the magnitudes of $\|f\|_{\bar{\Omega},\infty}$ and the boundary values $|u_\varepsilon(0)|$, $|u_\varepsilon(1)|$. These monontonicity-related restrictions are introduced at appropriate locations ((3.5) and (6.2)) in this paper. A representative example of the possible solutions to (1.1) is given in Figure 1, which illustrates the presence of an interior layer that steepens as $\varepsilon \to 0$.

A linear version of (1.1) was studied in [3], where a parameter-uniform numerical method based on a suitably designed piecewise-uniform mesh was shown to be parameter-uniform of essentially first order for a linear convection-diffusion problem with discontinuous data. The methodology in [3] is extended in this paper to the quasilinear problem (1.1). Note that if $\|u_\varepsilon\|_{\bar{\Omega},\infty} < 1$ in (1.1), then $b_1(u) < 0$ and

$b_2(u) > 0$. For this particular sign pattern either side of the point of discontinuity, a strong interior layer will normally be present in the solution. Alternative sign patterns on the coefficient of the first derivative can result in a weak interior layer appearing in the solution. These alternative sign patterns are discussed in [3] in the case of a linear version of problem (1.1). In [8], a parameter-uniform method was analysed for a semilinear singularly perturbed problem with discontinuous data, whose solution contained an interior layer. In this case, it is relatively straightforward to establish the conditions both for existence of the continuous solution and for inverse-monotonicity of a linearization of the discrete finite difference operator. The conditions on the data, under which existence can be established for the quasilinear problem examined in the present paper are more intricate. Moreover, the analysis of the inverse-monotonicity property of a linearized finite difference operator is significantly more complicated in the case of the quasilinear problem.

The paper is organized in the following manner. First, we establish the existence of the solution, and its uniqueness and derive *a priori* estimates. To do this we utilize an asymptotic approach. We associate a set of left and right problems with problem (1.1). The left problem and right problem are defined to be: find $u_L(x;\gamma) \in C^2(\Omega^-)$ and $u_R(x;\gamma) \in C^2(\Omega^+)$ such that

$$\text{(1.2a)} \quad \varepsilon u_L'' + b_1(u_L)u_L' = f,\ x \in \Omega^- = (0,d), \quad u_L(0) = u_\varepsilon(0),\ u_L(d) = \gamma,$$
$$\text{(1.2b)} \quad \varepsilon u_R'' + b_2(u_R)u_R' = f,\ x \in \Omega^+ = (d,1), \quad u_R(d) = \gamma,\ u_R(1) = u_\varepsilon(1).$$

In the next section we identify a natural restiction (2.4) on the data, so that there exist unique regular components (see Theorem 2.3) of the solutions to the problems (1.2a) and (1.2b). The left regular component $v_L(x)$ is defined so that it satisfies the same differential equation as $u_L(x;\gamma)$ on the interval Ω^-, agrees with u_L at the left boundary $x = 0$, and the first two derivatives of $v_L(x)$ are bounded independently of ε. Exterior to the interior layer region, the solution of (1.1) approaches the left (and right) regular component on Ω^- (and Ω^+). The multi-valued discontinuous regular component v_ε of (1.1) is defined to be the left regular component on $\bar{\Omega}^-$ and the right regular component on $\bar{\Omega}^+$, respectively. In order for this regular component to be monotonically increasing, we impose a further condition (3.5) on the data. In §3, we first establish existence and uniqueness of u_L and u_R for certain ranges of γ. We then show in Theorem 3.3 that by assuming (3.5), a value γ^* for the parameter γ can be chosen so that $u_R'(d^+,\gamma^*) = u_L'(d^-,\gamma^*)$. This establishes the existence of a solution to problem (1.1). In §4, the continuous solution u_ε to problem (1.1) is written as a sum of the discontinuous regular component v_ε and a discontinuous singular component w_ε. Parameter-explicit bounds on the first three derivatives of these two components are established in Lemma 4.2. The magnitude of the singular component is negligible outside of a $O(\varepsilon \ln \varepsilon)$-neighbourhood of the point $x = d$.

Based on these a priori bounds, a fitted mesh is constructed in §5. A nonlinear finite difference method is introduced and the existence of a discrete solution is established using appropriate choices of discrete lower and upper solutions. The existence of a discrete regular component is also established in this section. In §6 the main result (Theorem 6.2) of the paper is given. This shows that the numerical method produces numerical approximations, which converge to the unique solution of the continuous problem (1.1). The rate of convergence is independent of the small parameter ε. The method of proof requires that a discrete linear operator (associated with the nonlinear difference operator) preserve inverse-monotonicity. This requirement imposes an additional constraint (6.2) on the data for problem (1.1). At the end of §6, the implications of this assumption are discussed.

Numerical results are given in §7 and the appendix (§8) deals with discrete comparison principles for related linear problems, which are used in §6 in the proof of the main convergence result.

Throughout the paper C denotes a generic constant that is independent of ε and the mesh parameters. We always use the pointwise maximum norm and denote it by $\|z\|_D = \max_{x\in D} |z(x)|$. For notational convenience, we will omit the subscript when $D = \bar{\Omega}$ and simply write $\|z\|$.

2. Existence and uniqueness of the regular component

The regular components v_L and v_R of any possible solutions to problems (1.2a) and (1.2b) are formally defined to be the solutions of the two boundary value problems

$$\varepsilon v_L'' + b_1(v_L)v_L' = f,\ x < d, \tag{2.1a}$$
$$v_L(0) = u_\varepsilon(0),\ v_L(d) = v_0(d^-) + \varepsilon v_1(d^-),$$
$$\varepsilon v_R'' + b_2(v_R)v_R' = f,\ x > d, \tag{2.1b}$$
$$v_R(d) = v_0(d^+) + \varepsilon v_1(d^+),\ v_R(1) = u_\varepsilon(1),$$

where v_0 and v_1 are solutions of the following nonlinear first order problems

$$b(x, v_0)v_0' = f,\ x \in \Omega^- \cup \Omega^+,\ v_0(0) = u_\varepsilon(0);\ v_0(1) = u_\varepsilon(1), \tag{2.2a}$$
$$\varepsilon v_0'' + b(x, v_0 + \varepsilon v_1)(v_0 + \varepsilon v_1)' = f,\ x \in \Omega^- \cup \Omega^+, \tag{2.2b}$$
$$v_1(0) = 0,\ v_1(1) = 0.$$

By simply integrating, we have that the reduced solution v_0 satisfies the quadratic equations

$$(1 - 0.5cu_\varepsilon(0))u_\varepsilon(0) + \delta_1 x = (1 - 0.5cv_0(x))v_0(x),\ x < d, \tag{2.3a}$$
$$(1 + 0.5cu_\varepsilon(1))u_\varepsilon(1) - \delta_2(1 - x) = (1 + 0.5cv_0(x))v_0(x),\ x > d. \tag{2.3b}$$

If we assume that

$$\delta_1 d < -u_\varepsilon(0) + 0.5cu_\varepsilon^2(0) \quad \text{and} \quad \delta_2(1 - d) < u_\varepsilon(1) + 0.5cu_\varepsilon^2(1), \tag{2.4}$$

then there exists a unique reduced solution $v_0 \in C^1(\Omega^-) \cup C^1(\Omega^+)$ with the property that $v_0(x) < 0,\ x \in \Omega^-$ and $v_0(x) > 0,\ x \in \Omega^+$. Note the following additional properties of the reduced solution v_0 when (2.4) is satisfied:

$$b_1(v_0)(x) < -1, x < d, \quad b_2(v_0)(x) > 1, x > d,$$
$$v_0'(x) > 0,\ x \neq d,$$
$$\delta_1 > v_0'(x) > \frac{\delta_1}{1 - cu_\varepsilon(0)},\ x < d,\ v_0'(x) > \frac{\delta_2}{1 + cu_\varepsilon(1)},\ x > d,$$
$$v_0''(x) = \frac{-c(v_0')^2}{b(x, v_0)}, \quad (d - x)v_0''(x) > 0, \quad x \neq d.$$

From these properties and the fact that

$$v_0(d^-) = \int_0^d v_0'(x)dx + u_\varepsilon(0), \quad v_0(d^+) = u_\varepsilon(1) - \int_d^1 v_0'(x)dx,$$

we deduce the following bound on the jump in the reduced solution at $x = d$,

$$(2.5)\qquad [v_0](d) := v_0(d^+) - v_0(d^-) < u_\varepsilon(1) - u_\varepsilon(0) - \left(\frac{\delta_1 d}{1 - cu_\varepsilon(0)} + \frac{\delta_2(1-d)}{1 + cu_\varepsilon(1)}\right).$$

Let us now examine the second term $v_1(x)$ in the expansion of the regular component $v(x)$. Note that $b(x, v_0 + \varepsilon v_1) - b(x, v_0) = c\varepsilon v_1$. Hence, on both Ω^- and Ω^+,

$$(2.6)\ b(x, v_0 + \varepsilon v_1)v_1'(x) + cv_0'v_1(x) = (b(x, v_0)v_1)' + (0.5c\varepsilon v_1^2)' \ = \ -v_0''(x).$$

On Ω^-, integrate this equation from $t = 0$ to $t = x$, and on Ω^+, integrate from $t = 1$ to $t = x$. This yields a quadratic equation in v_1 of the form

$$b(x, v_0)v_1 + 0.5c\varepsilon v_1^2 = \begin{cases} -\displaystyle\int_0^x v_0''(t)\ dt < 0, & x < d, \\ \displaystyle\int_x^1 v_0''(t)\ dt < 0, & x > d. \end{cases}$$

On each subdomain, we require ε to be sufficiently small so that

$$b_1^2(v_0) > 2c\varepsilon \int_0^x v_0''(t)\ dt,\ x \in \Omega^- \quad \text{and} \quad b_2^2(v_0) > -2c\varepsilon \int_x^1 v_0''(t)\ dt,\ x \in \Omega^+.$$

With this restriction on ε, there are two possible solutions v_1^s, v_1^b with $0 < v_1^s < v_1^b$, $x \in \Omega^-$ and $v_1^b < v_1^s < 0$, $x \in \Omega^+$. Define v_1 uniquely by setting $v_1 = v_1^s$. Consider a quadratic of the form $k(x) = 0.5l\varepsilon x^2 - mx + I$, where l, m, I are positive constants and ε is sufficiently small so that $m^2 > 2l\varepsilon I$. Then the minimum of k occurs at $x = m/(l\varepsilon)$. Since $k(2I/m) < 0$, we note that the smallest root of $k(x) = 0$ is smaller than $2I/m$. Hence, on the intervals Ω^- and Ω^+ select the smallest roots such that

$$\frac{-2\int_0^x v_0''(t)\ dt}{b_1(v_0)} > v_1 > 0, \quad x \in \Omega^-, \ \ 0 > v_1 > \frac{2\int_x^1 v_0''(t)\ dt}{b_2(v_0)}, \quad x \in \Omega^+.$$

This yields a unique v_1 (with $v_1'(x) > 0$), which is bounded independently of ε and from (2.6) it follows that

$$|v_1'(x)| \le C, \quad |v_1''(x)| \le C.$$

To establish the existence and uniqueness of the regular component, we employ the technique of upper and lower solutions.

Definition 2.1. A function $\alpha \in C^1(\Omega^-)$ is a *lower solution* of problem (1.2a) if

$$(2.7)\qquad \varepsilon\alpha'' + b_1(\alpha)\alpha' \ge f,\ x < d \quad \text{and} \quad \alpha(0) \le u_\varepsilon(0),\ \alpha(d) \le \gamma.$$

An *upper solution* β is defined in an analogous fashion, with all inequalities reversed. Consider the general quasilinear problem

$$\varepsilon y'' = g(x, y, y'), \quad x \in J = (a_1, a_2).$$

Let $g \in C[J \times \mathbb{R} \times \mathbb{R}; \mathbb{R}]$ and $\alpha, \beta \in C[J, \mathbb{R}]$ with $\alpha(x) \le \beta(x), x \in J$. Suppose that for $x \in J, \alpha(x) \le y(x) \le \beta(x)$,

$$|g(x, y, y')| \le \Psi(|y'|),$$

where $\Psi \in C[[0, \infty), (0, \infty)]]$. If

$$\int_\lambda^\infty \frac{s}{\Psi(s)}\ ds > \max_{x \in J} \beta(x) - \min_{x \in J} \alpha(x),$$

where $\lambda(a_2-a_1) = \max\{|\alpha(a_1)-\beta(a_2)|, |\alpha(a_2)-\beta(a_1)|\}$, then we say that g satisfies a Nagumo condition on J relative to the lower and upper solutions [1].

The nonlinearity in (1.2a) (and (1.2b)) satisfies a Nagumo condition for any bounded α and β, since we can take $\Psi(x) = \|f\| + \gamma x$, $\gamma := \sup_{\alpha\leq y\leq\beta} |b_1(y)|$. Thus we can cite the following existence result.

Lemma 2.2 ([1, page 31]). *If $\alpha, \beta \in C^1(\Omega^-, \mathbb{R})$ are lower and upper solutions for the problem* (1.2a) *and $\alpha(x) \leq \beta(x)$, $\forall x \in \bar{\Omega}^-$, then there exists a solution $u_L \in C^2(\Omega^-, \mathbb{R})$ to* (1.2a) *and $\alpha(x) \leq u_L(x) \leq \beta(x)$, $\forall x \in \bar{\Omega}^-$.*

Hence, to establish existence of a regular component v_L on Ω^-, it suffices to construct lower and upper solutions.

Theorem 2.3. *Assume* (2.4). *There is a unique regular solution to* (2.1a) *and $v_L \in C^\infty(\Omega^-, [u_\varepsilon(0), 0))$, with $v_L(x) \geq v_0(x)$.*

Proof. Note that, by assuming (2.4), $v_0(d^-) < 0$ and for ε sufficiently small,

$$v_0(d^-) \leq v_L(d) = v_0(d^-) + \varepsilon v_1(d^-) < 0.$$

Since $v_L(d) < 0$ on Ω^-, we can use the lower and upper solutions $\alpha(x) = v_0(x)$ and $d\beta(x) = -u_\varepsilon(0)(x-d)$ to establish existence of a regular solution v_L satisfying (2.1a) and (2.4) on Ω^-. Suppose $v_1, v_2 \in C^2\big((0,d), [u_\varepsilon(0), 0)\big)$ are two regular solutions of (2.1a) and let $\psi = v_1 - v_2$. Then

$$\varepsilon\psi'' = \psi' + 0.5c(v_2^2 - v_1^2)', \quad \psi(0) = \psi(d) = 0.$$

Integrating from $x = 0$ to $x = t$ yields $\varepsilon\psi'(t) - (1 - 0.5c(v_1+v_2)(t))\psi(t) = \varepsilon\psi'(0)$. Since $\psi(d) = 0$, $\psi(t) \equiv 0$. Hence the solution of (2.1a) is unique. $\square$

There is an analogous result for the existence and uniqueness of a solution $v_R \in C^\infty(\Omega^+, (0, u_\varepsilon(1)])$ of (2.1b). Define the regular component of any solution to (1.1) to be the multi-valued discontinuous function

$$v_\varepsilon(x) := \begin{cases} v_L(x), & x \leq d, \\ v_R(x), & x \geq d, \end{cases} \tag{2.8}$$

where v_L is the solution of (2.1a) and v_R is the solution of (2.1b).

3. Existence and uniqueness of the continuous solution

In this section we establish the existence of $u_L(x;\gamma)$ (and $u_R(x;\gamma)$) for a certain range of γ. Under additional assumptions on the data, we will show that $u_L(x;\gamma)$ and $u_R(x;\gamma)$ both exist for a common range of γ. To this end, we define the barrier functions $\chi_L(x;\gamma)$, $\chi_R(x;\gamma)$ as the respective solutions of the boundary value problems

$$\varepsilon\chi_L'' + (-1+c\gamma)\chi_L' = -\delta_1, \quad \chi_L(0) = u_\varepsilon(0),\ \chi_L(d) = \gamma, \tag{3.1a}$$

$$\varepsilon\chi_R'' + (1+c\gamma)\chi_R' = \delta_2, \quad \chi_R(d) = \gamma,\ \chi_R(1) = u_\varepsilon(1). \tag{3.1b}$$

Lemma 3.1. *For all $\gamma \in [v_L(d), \frac{1}{c})$ such that*

$$\delta_1 d \leq (1 - c\gamma)(\gamma - u_\varepsilon(0)), \tag{3.2}$$

problem (1.2a), (2.4) *has a unique solution $u_L(x;\gamma) \in C^2\big((0,d), [u_\varepsilon(0), \gamma]\big)$ with the property that*

$$v_L(x) \leq u_L(x;\gamma) \leq \chi_L(x;\gamma), \quad x \in [0,d].$$

For all $\gamma \in \left(-\frac{1}{c}, v_R(d)\right]$ *such that*

$$\delta_2(1-d) \leq (1+c\gamma)(u_\varepsilon(1)-\gamma), \tag{3.3}$$

problem (1.2b), (2.4) *has a unique solution* $u_R(x;\gamma) \in C^2\big((0,d),[\gamma, u_\varepsilon(1)]\big)$ *with the property that*

$$\chi_R(x;\gamma) \leq u_R(x;\gamma) \leq v_R(x), \quad x \in [d,1].$$

Proof. Note that

$$\chi_L(x;\gamma) = \frac{\delta_1}{1-c\gamma}x + u_\varepsilon(0) + K\psi(x;\gamma),$$

where

$$\varepsilon\psi'' - (1-c\gamma)\psi' = 0,\ \psi(0)=0,\ \psi(d)=1, \quad K = \frac{(1-c\gamma)(\gamma - u_\varepsilon(0)) - \delta_1 d}{1-c\gamma}.$$

Also by (3.2) $K \geq 0$ and so $\chi_L' > 0$. Hence

$$\begin{aligned} \varepsilon\chi_L'' + b_1(\chi_L)\chi_L' &= -\delta_1 + (b_1(\chi_L) - (-1+c\gamma))\chi_L' \\ &= -\delta_1 + c(\chi_L - \gamma)\chi_L' \leq -\delta_1. \end{aligned}$$

An analagous argument is used to establish the existence of $u_R(x;\gamma)$. □

If

$$\delta_1 d \leq -u_\varepsilon(0) + cv_R(d)\left(\frac{1}{c} + u_\varepsilon(0) - v_R(d)\right) \tag{3.4a}$$

and

$$\delta_2(1-d) \leq u_\varepsilon(1) - cv_L(d)\left(\frac{1}{c} - u_\varepsilon(1) + v_L(d)\right), \tag{3.4b}$$

then by the previous lemma $u_L(x;v_R(d))$ and $u_R(x;v_L(d))$ both exist. Hence to guarantee the existence of a continuous solution $u(x;\gamma)$ defined over the entire interval $[0,1]$ for all $\gamma \in [v_L(d), v_R(d)]$, we are required to restrict the data of problem (1.1). Hence we are led to the following assumption.

Assumption 1. Assume that the problem data for problem (1.1) are such that

$$\delta_1 d < -u_\varepsilon(0), \quad \delta_2(1-d) < u_\varepsilon(1) \tag{3.5a}$$

and

$$u_\varepsilon(1) - u_\varepsilon(0) < 1/c + \min\left\{\frac{\delta_1 d}{1-cu_\varepsilon(0)}, \frac{\delta_2(1-d)}{1+cu_\varepsilon(1)}\right\}. \tag{3.5b}$$

Note that (3.5) implies (2.4). By the properties of $v_0(x)$ established in §2, it follows from (3.5) that, for ε sufficiently small,

$$0 > v_L(d) > -\frac{1-cu_\varepsilon(1)}{c} \quad \text{and} \quad 0 < v_R(d) < \frac{1+cu_\varepsilon(0)}{c}.$$

The assumption (3.5) suffices for the inequalities (3.4) to hold true and consequently for $u_L(x;\gamma)$ and $u_R(x;\gamma)$ to exist for all $\gamma \in [v_L(d), v_R(d)]$. In the next lemma we establish that $u_L'(x;\gamma)$ and $u_R'(x;\gamma)$ depend continuously on the parameter γ.

Lemma 3.2. *Assuming* (3.5), *for all* $\gamma_1, \gamma_2 \in [v_L(d), v_R(d)]$,

$$\begin{aligned} \varepsilon|u_L'(x;\gamma_1) - u_L'(x;\gamma_2)| &\leq C|\gamma_1 - \gamma_2|, \quad x \in (0,d), \\ \varepsilon|u_R'(x;\gamma_1) - u_R'(x;\gamma_2)| &\leq C|\gamma_1 - \gamma_2|, \quad x \in (d,1). \end{aligned}$$

Proof. Let $G(x;\gamma_1,\gamma_2) = u_L(x;\gamma_1) - u_L(x;\gamma_2)$. Note that

$$\varepsilon G'' + \Big(-1 + \frac{c}{2}(u_L(x;\gamma_1) + u_L(x;\gamma_2))\Big) G' = 0, \quad G(0) = 0,\ G(d) = \gamma_1 - \gamma_2,$$

and $(-1+\frac{c}{2}(u_L(x;\gamma_1)+u_L(x;\gamma_2)) \le -1+0.5c(\gamma_1+\gamma_2) \le -1+cv_R(d) \le cu_\varepsilon(0) < 0$. It follows that $\varepsilon|G'| \le C|\gamma_1 - \gamma_2|$. □

We now state a central result in this paper.

Theorem 3.3. *Assuming* (3.5), *the nonlinear problem* (1.1) *has a unique solution* $u_\varepsilon \in C^1((0,1),(u_\varepsilon(0),u_\varepsilon(1))$. *Moreover,*

$$v_L(x) \le u_\varepsilon(x) \le \chi_L(x;v_R(d)),\ x \le d, \quad \chi_R(x;v_L(d)) \le u_\varepsilon(x) \le v_R(x),\ x \ge d,$$

where χ_L, χ_R *are defined in* (3.1).

Proof. For all $x \in \Omega^-$,

$$\int_{t=0}^{x} (f - b_1(u_L)u_L')(t)\ dt = (u_L(x) - u_\varepsilon(0))\big(1 - 0.5c(u_L(x) + u_\varepsilon(0))\big) - \delta_1 x.$$

Integrating (1.2a), from 0 to x, yields

$$\varepsilon u_L'(x;\gamma) = \varepsilon u_L'(0;\gamma) + (u_L(x;\gamma) - u_\varepsilon(0))\big(1 - 0.5c(u_L(x;\gamma) + u_\varepsilon(0))\big) - \delta_1 x.$$

By the Mean Value Theorem, for some $z \in (0,\varepsilon)$, $\varepsilon < d$, $\varepsilon u_L'(z;\gamma) = u_L(\varepsilon;\gamma) - u_\varepsilon(0)$. Since $c\gamma < 1$ and (3.2), using the lower and upper solutions given in Lemma 3.1, we deduce that, for all $\gamma \in [v_L(d), v_R(d)]$,

$$0 \le u_L(z;\gamma) - u_\varepsilon(0) \le C\varepsilon, \quad 0 \le u_L'(0;\gamma) \le C.$$

For all $x \in \Omega^+$,

$$\int_{t=x}^{1} (f - b_2(u_R)u_R')(t)\ dt = \delta_2(1-x) + (u_R(x) - u_\varepsilon(1))\big(1 + 0.5c(u_R(x) + u_\varepsilon(1))\big).$$

Integrating (1.2b) from x to 1 yields

$$\varepsilon u_R'(x;\gamma) = \varepsilon u'(1;\gamma) - \delta_2(1-x) + (u_\varepsilon(1) - u_R(x;\gamma))\big(1 + 0.5c(u_R(x;\gamma) + u_\varepsilon(1))\big).$$

Since $c\gamma > -1$, using (3.2) and the lower and upper solutions given in Lemma 3.1, we deduce that, for all $\gamma \in [v_L(d), v_R(d)]$,

$$0 \le u_R'(1;\gamma) \le C.$$

We wish to establish the existence of a $\gamma^* = u_L(d) = u_R(d)$ such that $u_L'(d^-;\gamma^*) = u_R'(d^+;\gamma^*)$ and $-1 < c\gamma^* < 1$. This is equivalent to finding a γ^* such that $-1 < c\gamma^* < 1$ and

$$\begin{aligned} &\varepsilon u_L'(0;\gamma^*) + (\gamma^* - u_\varepsilon(0))\big(1 - 0.5c(\gamma^* + u_\varepsilon(0))\big) - \delta_1 d \\ &= \varepsilon u_R'(1;\gamma^*) - \delta_2(1-d) + (u_\varepsilon(1) - \gamma^*)\big(1 + 0.5c(\gamma^* + u_\varepsilon(1))\big). \end{aligned}$$

Rearranging, gives

$$2\gamma^* = \varepsilon u_R'(1;\gamma^*) - \varepsilon u_L'(0;\gamma^*) + \delta_1 d + u_\varepsilon(0) - 0.5cu_\varepsilon^2(0) + u_\varepsilon(1) + 0.5cu_\varepsilon^2(1) - \delta_2(1-d).$$

By (2.3), this further simplifies to

$$2\gamma^* = \Gamma + \varepsilon u_R'(1;\gamma^*) - \varepsilon u_L'(0;\gamma^*), \tag{3.6}$$

where $\Gamma := v_0(d^+) + v_0(d^-) + 0.5cv_0^2(d^+) - 0.5cv_0^2(d^-)$. Define the function

$$H(\gamma) := \Gamma + \varepsilon u_R'(1;\gamma) - \varepsilon u_L'(0;\gamma) - 2\gamma.$$

For all $\gamma \in [v_L(d), v_R(d)]$,

$$|\varepsilon u_R'(1;\gamma) - \varepsilon u_L'(0;\gamma)| \le C\varepsilon.$$

Note that $v_L(d) < 0 < v_R(d)$ and so, for ε sufficiently small, $H(v_L(d)) > 0 > H(v_R(d))$. By Lemma 3.2, H is a continuous function of γ, so there exists a $\gamma^* \in [v_L(d), v_R(d)]$, where $H(\gamma^*) = 0$. Hence we have established the existence of a solution $u_\varepsilon \in C^1((0,1),(u_\varepsilon(0),u_\varepsilon(1))$ to problem (1.1), (3.5).

Let u^+, u^- be two solutions of problem (1.1). The difference in these two solutions is $y := u^+ - u^- \in C^1((0,1),(u_\varepsilon(0),u_\varepsilon(1))$ and solves the problem

$$\varepsilon y'' + b(x,u^+)(u^+)' - b(x,u^-)(u^-)' = 0, \quad y(0) = y(1) = 0.$$

Integrate over Ω^- and over Ω^+ to get

$$\varepsilon y' - y + \frac{1}{2}c(u^+ + u^-)y = \varepsilon y'(0), \ x \in \Omega^-,$$
$$\varepsilon y' + y + \frac{1}{2}c(u^+ + u^-)y = \varepsilon y'(1), \ x \in \Omega^+.$$

Using integrating factors and integrating again, we have that

$$y = y'(0)e^{\frac{p(x)}{\varepsilon}} \int_0^x e^{\frac{-p(t)}{\varepsilon}}\, dt, \quad p(x) = \int_0^x 1 - \frac{1}{2}c(u^+ + u^-)\, dt, \ x \in \Omega^-,$$
$$y = -y'(1)e^{\frac{-q(x)}{\varepsilon}} \int_x^1 e^{\frac{q(t)}{\varepsilon}}\, dt, \quad q(x) = \int_x^1 1 + \frac{1}{2}c(u^+ + u^-)\, dt, \ x \in \Omega^-.$$

Note that y is continuous at $x = d$, so from above $y'(0)y'(1) \le 0$. If $y'(0) = 0$, then $y = 0$ for $x \in \Omega^-$, which implies that $y(d) = 0$. This, in turn, implies that $y = 0, \ x \in \Omega$. Without loss of generality, let us assume that $y'(0) > 0$. From the expression for y above, $y > 0$, for $x \in \Omega^-$. Also, from the expression for y' and the fact that $1 - \frac{1}{2}c(u^+ + u^-) > 0$, we deduce that $y' > 0$, for $x \in \Omega^-$. Also, if $y'(0) > 0$, then $y'(1) < 0$. Repeating the above argument, we deduce that $y'(x) < 0, x \in \Omega^+$. Hence, the maximum value of y occurs at $x = d$. Subtracting the two expressions for the derivative of y at $x = d$, yields $2y(d) = \varepsilon(y'(1) - y'(0)) < 0$, which is a contradiction. Hence $y'(0) = 0$, which implies that $y = 0$. This establishes the uniqueness of u_ε. □

Remark 3.4. Note that for the solution to (1.1), (3.5) we have that

$$b_1(u_\varepsilon) \le b_1(v_R(d)) \le -1 + cv_R(d) < cu_\varepsilon(0) < 0, \quad x \le d,$$
$$b_2(u_\varepsilon) \ge b_2(v_L(d)) \ge 1 + cv_L(d) > cu_\varepsilon(1) > 0, \quad x \ge d.$$

Recall that we also have $|b_1(u_\varepsilon)| > 1 - cu_\varepsilon(1), x \le d$ and $b_2(u_\varepsilon) > 1 + cu_\varepsilon(0), x \ge d$. Combining these we get

$$|b_1(u_\varepsilon)| > \theta_1 := \max\{-cu_\varepsilon(0), 1 - cu_\varepsilon(1)\}, \quad x \le d, \tag{3.7a}$$
$$b_2(u_\varepsilon) > \theta_2 := \max\{cu_\varepsilon(1), 1 + cu_\varepsilon(0)\}, \quad x \ge d. \tag{3.7b}$$

In the next lemma we state parameter-explicit pointwise estimates on the derivatives of the solution to (1.1), (3.5).

Lemma 3.5. *Let u_ε be the solution of* (1.1), (3.5), *then, for all* $1 \le k \le 3$,

$$|u_\varepsilon^{(k)}(x)| \le C\varepsilon^{-k}, \quad x \in \Omega^- \cup \Omega^+.$$

Proof. Use the argument from the proof of the previous lemma to establish that $|\varepsilon u_\varepsilon'| \le C$. Then use the differential equation (1.1a) to get the bounds on the second and third derivatives of u_ε. □

4. A priori bounds on the singular component

Since the solution u_ε of (1.1) and the regular component v_ε defined in (2.8) are uniquely defined, we can define the discontinuous singular component w_ε implicitly by $u_\varepsilon = w_\varepsilon + v_\varepsilon$ and

$$\varepsilon u_\varepsilon'' + b(x,u_\varepsilon)u_\varepsilon' = f, \quad x \ne d, \tag{4.1a}$$

$$u_\varepsilon \in C^1(0,1), \qquad u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B. \tag{4.1b}$$

Since u_ε and v_ε are unique, we have that $\|w_\varepsilon\| = \|u_\varepsilon - v_\varepsilon\| \le \|u_\varepsilon\| + \|v_\varepsilon\| \le C$, and the singular component w_ε is the solution of

$$\varepsilon w_\varepsilon'' + b(x,u_\varepsilon)w_\varepsilon' + (cv_\varepsilon')w_\varepsilon = 0,\ x \ne d, \quad w_\varepsilon(0) = w_\varepsilon(1) = 0,$$
$$[w_\varepsilon](d) = -[v_\varepsilon](d), \quad [w_\varepsilon'](d) = -[v_\varepsilon'](d), \quad [\omega](d) := \omega(d+) - \omega(d-).$$

Let L_ε denote the linear differential operator, which is defined as

$$L_\varepsilon\omega := \varepsilon\omega'' + a(x)\omega' + b(x)\omega,$$

where

$$a(x) \le -\alpha_1 < 0,\ x < d, \quad a(x) \ge \alpha_2 > 0,\ x > d,$$

and, for $\alpha = \min\{\alpha_1, \alpha_2\}$, $\alpha^2 - 4\varepsilon b > 0$, $\forall x \ne d$.

The differential operator L_ε satisfies the following comparison principle.

Lemma 4.1. *Suppose that a function $\omega \in C^0(\bar\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies $\omega(0) \le 0$, $\omega(1) \le 0$, $[\omega'](d) \ge 0$, and $L_\varepsilon\omega(x) \ge 0$, for all $x \in \Omega^- \cup \Omega^+$, then $\omega(x) \le 0$, for all $x \in \bar\Omega$.*

Proof. Follow the proof of the corresponding result in [3], but include the zero-order term in the proof. Introduce the function $v(x)$, defined by

$$\omega(x) = e^{-\alpha(x)|x-d|/(2\varepsilon)}v(x),$$

where $\alpha(x) = \alpha_1, x < d, \alpha(x) = \alpha_2, x > d$. Hence, for $x \in \Omega^-$,

$$L_\varepsilon\omega = e^{-\theta(x)|x-d|/(2\varepsilon)}\left(\varepsilon v'' + (a+\alpha_1)v' + (\frac{\alpha_1^2}{4\varepsilon} + \frac{a\alpha_1}{2\varepsilon} + b)v\right),$$

and for $x \in \Omega^+$,

$$L_\varepsilon\omega = e^{-\alpha(x)|x-d|/(2\varepsilon)}\left(\varepsilon v'' + (a-\alpha_2)v' + (\frac{\alpha_2^2}{4\varepsilon} - \frac{a\alpha_2}{2\varepsilon} + b)v\right).$$

Assume that $\max_{\bar\Omega} v = v(q) > 0$. With the above assumption on the boundary values, either $q \in \Omega^- \cup \Omega^+$ or $q = d$. If $q \in \Omega^-$, then

$$L_\varepsilon\omega(q) = e^{-\alpha_1(d-q)/(2\varepsilon)}\left(\varepsilon v''(q) + (b - \frac{\alpha_1^2}{4\varepsilon})v(q)\right) < 0,$$

which is a contradiction. If $q \in \Omega^+$, then an analogous argument also leads to a contradiction. The only possibility remaining is that $q = d$. Note that $[v](d) = [\omega](d) = 0$ and $[\omega'](d) = [v'](d) - \frac{\alpha_1+\alpha_2}{2\varepsilon}v(d)$. Since d is where v takes its maximum value, $v'(d-) \ge 0$, $v'(d+) \le 0$, which implies that $[v'](d) \le 0$. This implies that $[\omega'](d) < 0$, which is a contradiction. □

Lemma 4.2. *Assume* (3.5). *For each integer k, satisfying $1 \le k \le 3$, the solutions v_ε and w_ε of* (2.8) *and* (4.1), *respectively, satisfy the following bounds:*

$$\|v_\varepsilon\| \le C, \quad \|v_\varepsilon^{(k)}\|_{\Omega^- \cup \Omega^+} \le C(1+\varepsilon^{2-k}),$$

$$|[v_\varepsilon](d)| \le C, |[v_\varepsilon'](d)| \le C, |[v_\varepsilon''](d)| \le C,$$

$$|w_\varepsilon^{(k)}(x)| \le \begin{cases} C\varepsilon^{-k}e^{-(d-x)\theta_1/\varepsilon}, & x \in \Omega^-, \\ C\varepsilon^{-k}e^{-(x-d)\theta_2/\varepsilon}, & x \in \Omega^+, \end{cases}$$

where C is a constant independent of ε, and θ_1, θ_2 are given in (3.7).

Proof. Define v_2 to be such that $v_\varepsilon := v_0 + \varepsilon v_1 + \varepsilon^2 v_2$. From above such a function exists and is unique. Note that v_2 is the solution of

$$\varepsilon(v_0'' + \varepsilon v_1'' + \varepsilon^2 v_2'') + b(x, v_\varepsilon)(v_0' + \varepsilon v_1' + \varepsilon^2 v_2') = f = b(x, v_0)(v_0)', \quad x \neq d.$$

Hence

$$\varepsilon v'' + \varepsilon b(x, v_\varepsilon)(v_1' + \varepsilon v_2') = -c\varepsilon(v_1 + \varepsilon v_2)v_0',$$

which can be written in the form

$$v'' + b(x, v_\varepsilon)(v_1' + \varepsilon v_2') = -c(v_1 + \varepsilon v_2)v_0'.$$

From the definition of v_1, we have that

$$b(x,v)v_1' = \big(b(x,v) - b(x, v_0 + \varepsilon v_1)\big)v_1' - cv_0'v_1 - v_0''.$$

Inserting this into the equation above and simplifying, shows that v_2 satisfies the following problem:

$$\varepsilon v_2'' + b(x,v)v_2' + c(v_0' + \varepsilon v_1')v_2 = -v_1'', \quad x \neq d, \tag{4.2a}$$

$$v_2(0) = v_2(d) = v_2(1) = 0. \tag{4.2b}$$

Note that $v_0' + \varepsilon v_1' > 0$ and, for ε sufficiently small, $b^2(x,v) - 4c\varepsilon(v_0' + \varepsilon v_1') > 0$. We rewrite (4.2a) in the form

$$\varepsilon v_2'' = g(x, v_2) := -\big(b(x,v)v_2' + c(v_0' + \varepsilon v_1')v_2 + v_1''\big).$$

Define $M_1 := \|v_1''\|$ and $\beta_1 := \min c(v_0' + \varepsilon v_1') > 0$. Check that $\alpha(x), \beta(x)$ defined by $-\beta_1\alpha(x) = \beta_1\beta(x) = M_1$ are lower and upper solutions. Thus

$$\|v_2\| \le \frac{M_1}{\beta_1}.$$

Note that

$$\varepsilon v_2'' + b(x,v)v_2' = g_1 := -v_1'' - c(v_0' + \varepsilon v_1')v_2,$$

which implies that $\|v_2\| \le Cx\|g_1\|$, $x < d$. Thus $|v_2'(0)| \le C$, and using integration we have $|v_2'(x)| \le C$, $x < d$. The bounds on the derivatives of v_2 follow.

Now we estimate the singular term. Note that

$$\varepsilon u_\varepsilon'' + b(x, u_\varepsilon)u_\varepsilon' = f = \varepsilon v_\varepsilon'' + b(x, v_\varepsilon)v_\varepsilon', \quad x \neq d.$$

Hence

$$\varepsilon w_\varepsilon'' + (b(x, u_\varepsilon) - b(x, v_\varepsilon))v_\varepsilon' + b(x, u_\varepsilon)w_\varepsilon' = \varepsilon w_\varepsilon'' + b(u_\varepsilon)w_\varepsilon' + (cv_\varepsilon')w_\varepsilon = 0, \quad x \neq d,$$

and $w_\varepsilon(0) = 0$, $w_\varepsilon(1) = 0$, $w_\varepsilon(d^-) = u_\varepsilon(d^-) - v_\varepsilon(d^-)$, $w_\varepsilon(d^+) = u_\varepsilon(d^+) - v_\varepsilon(d^+)$. Choose ε sufficiently small so that $b^2(x, u_\varepsilon) - 4\varepsilon c v_\varepsilon' > 0$. Then we can apply the arguments from the linear problem [3] and Lemma 4.1 to get bounds on w_ε and its derivatives separately on Ω^- and Ω^+. Note that we require ε sufficiently small so that the barrier function

$$B(x) = \begin{cases} Ce^{-(d-x)\theta_1/\varepsilon}, & x \in \Omega^-, \\ Ce^{-(x-d)\theta_2/\varepsilon}, & x \in \Omega^+, \end{cases}$$

satisfies the inequalities

$$\begin{aligned} \varepsilon B''(x) + b_1(u_\varepsilon)B' + cv_\varepsilon' B &\le \frac{1}{\varepsilon}\{\theta_1(\theta_1 - |b_1(u_\varepsilon)|) + \varepsilon(cv_\varepsilon')\}B < 0, \quad x \in \Omega^-, \\ \varepsilon B''(x) + b_2(u_\varepsilon)B' + cv_\varepsilon' B &\le \frac{1}{\varepsilon}\{\theta_2(\theta_2 - b_2(u_\varepsilon)) + \varepsilon(cv_\varepsilon')\}B < 0, \quad x \in \Omega^+. \end{aligned}$$

□

5. Existence of discrete solutions

The domain $\overline{\Omega}$ is subdivided into four subintervals

$$[0, d-\sigma_1] \cup [d-\sigma_1, d] \cup [d, d+\sigma_2] \cup [d+\sigma_2, 1]. \tag{5.1a}$$

The transition points σ_1 and σ_2 are defined by

$$\sigma_1 = \min\left\{\frac{d}{2},\ 2\frac{\varepsilon}{\theta_1}\ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1-d}{2},\ 2\frac{\varepsilon}{\theta_2}\ln N\right\}, \tag{5.1b}$$

where $\theta_1 = \max\{-cu_\varepsilon(0), 1 - cu_\varepsilon(1)\}$ and $\theta_2 = \max\{cu_\varepsilon(1), 1 + cu_\varepsilon(0)\}$ as defined in (3.7). On each of the four subintervals a uniform mesh with $\frac{N}{4}$ mesh-intervals is placed. The mesh points are denoted by $\overline{\Omega}_\varepsilon^N = \{x_i\}_0^N$, where $x_0 = 0$, $x_N = 1$, $x_{N/2} = d$. The fitted mesh method for problem (1.1) is: find a mesh function U_ε such that

$$\begin{aligned} &(5.2a) & \varepsilon\delta^2 U_\varepsilon(x_i) + b(x_i, U_\varepsilon(x_i))DU_\varepsilon(x_i) &= f(x_i) \quad \text{for all} \quad x_i \in \Omega_\varepsilon^N, \\ &(5.2b) & U_\varepsilon(0) = u_\varepsilon(0), &\qquad U_\varepsilon(1) = u_\varepsilon(1), \\ &(5.2c) & D^- U_\varepsilon(d) &= D^+ U_\varepsilon(d), \end{aligned}$$

where

$$\delta^2 Z_i = \frac{D^+ Z_i - D^- Z_i}{(x_{i+1} - x_{i-1})/2} \quad \text{and} \quad DZ_i = \begin{cases} D^- Z_i, & i < N/2, \\ D^+ Z_i, & i > N/2. \end{cases}$$

Here D^+ and D^- are the standard forward and backward finite difference operators, respectively. This is a nonlinear finite difference scheme.

Let $G : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be a mapping associated with this finite difference scheme. For mesh function Y, we have an associated vector $Y \in \mathbb{R}^{N+1}$, where $Y_i = Y(x_i)$. Let

$$(GY)_i = \begin{cases} -Y(0), \quad i = 0, \\ \varepsilon\delta^2 Y_i + b(x_i, Y_i)DY_i, \quad i \ne N/2, 0 < i < N, \\ \varepsilon\delta^2 Y_i, \quad i = N/2, \\ -Y(1), \quad i = N. \end{cases}$$

We also define a vector F by

$$F_i = \begin{cases} A, 0, B, \quad i = 0, N/2, N, \\ f(x_i), \quad \text{otherwise.} \end{cases}$$

The finite difference scheme (5.2) can then be written in the form $GU_\varepsilon = F$.

Definition 5.1. Given any vector $H \in \mathbb{R}^{N+1}$, a *lower mesh solution* V for the problem $GW = H$ is a mesh function, which satisfies $GV \ge H$.

There is an analogous definition for an upper mesh solution to $GW = H$.

Theorem 5.2. *If Φ and Ψ are, respectively, lower and upper mesh solutions for the problem $GW = H$, with the additional properties that $-1 < c\Phi < 1, -1 < c\Psi < 1$, and $\Phi(x_i) \le \Psi(x_i)$, $\forall x_i \in \bar{\Omega}^N$, then there exists a solution to $GW = H$ and $\Phi(x_i) \le W(x_i) \le \Psi(x_i)$, $\forall x_i \in \bar{\Omega}^N$.*

Proof. We follow the argument from Lorentz [11]. Let Φ_1, Φ_2 be two lower mesh functions. Define the mesh function Φ_3 by $\Phi_3(x_i) := \max\{\Phi_1(x_i), \Phi_2(x_i)\}$. At some point x_j, we assume, w.l.o.g., that $\Phi_3(x_j) = \Phi_1(x_j)$. Note that $-\Phi_3(x_i) \le -\Phi_1(x_i)$, $\forall x_i$. For any x_j,

$$\begin{aligned}
\varepsilon\delta^2\Phi_3(x_j) + b(x_j, \Phi_3)D\Phi_3(x_j) &\ge \varepsilon\delta^2\Phi_1(x_j) + b(x_j, \Phi_1)D\Phi_1(x_j) \\
&\ge H(x_j), \quad x_j \ne d, \\
\varepsilon\delta^2\Phi_3(x_j) \ge \varepsilon\delta^2\Phi_1(x_j) &\ge H(d), \quad x_j = d, \\
\Phi_3(0) \le H(0), \qquad & \Phi_3(1) \le H(1).
\end{aligned}$$

Then Φ_3 is also a lower mesh solution. Let $L = \{\phi : G\phi \ge H, \Phi \le \phi \le \Psi\}$ be the set of all possible lower mesh solutions. Define $U(x_i) := \sup_{\phi \in L}\{\phi(x_i)\}$. First note that $U \in L$ exists and $GU \ge H$. Assume that we do not have equality, then there exists some j such that $GU(x_j) > H(x_j)$. If $U \ne \Psi$, construct a new vector $Y = U + \gamma\delta_{i,j}$, $\gamma > 0$. Then γ can be chosen sufficiently small such that

$$GY = GU + c(Y - U)DU \ge H.$$

Hence, $Y \in L$, $U < Y$, which is a contradiction. $\square$

Define the mesh functions V_L and V_R to be the solutions of the following discrete nonlinear problems:

$$L^N_{\text{left}}V_L := \left\{\varepsilon\delta^2 + b_1(V_L)D^-\right\}V_L(x_i) = -\delta_1, \; x_i \in \Omega^N_\varepsilon \cap \Omega^-, \tag{5.3a}$$

$$V_L(0) = v_\varepsilon(0), \; V_L(d) = v_\varepsilon(d^-), \tag{5.3b}$$

$$L^N_{\text{right}}V_R := \left\{\varepsilon\delta^2 + b_2(V_R)D^+\right\}V_R(x_i) = \delta_2, \; x_i \in \Omega^N_\varepsilon \cap \Omega^+, \tag{5.3c}$$

$$V_R(1) = v_\varepsilon(1), \quad V_R(d) = v_\varepsilon(d^+). \tag{5.3d}$$

In an analogous fashion to Theorem 5.2 we have the following

Theorem 5.3. *If Φ and Ψ are two mesh functions such that*

$$\begin{aligned}
\Phi(0) \le V_L(0) \le \Psi(0), \qquad & \Phi(d) \le V_L(d) \le \Psi(d), \\
\Phi(d) \le V_R(d) \le \Psi(d), \qquad & \Phi(1) \le V_R(1) \le \Psi(1), \\
L^N_{\text{left}}\Phi \ge L^N_{\text{left}}V_L \ge L^N_{\text{left}}\Psi, \qquad & L^N_{\text{right}}\Phi \ge L^N_{\text{right}}V_R \ge L^N_{\text{right}}\Psi
\end{aligned}$$

with the additional properties that

$$-1 < c\Phi < 1, -1 < c\Psi < 1, \qquad \Phi(x_i) \le \Psi(x_i), \; \forall x_i \in \bar{\Omega}^N,$$

then there exists a solution to (5.3) and

$$\begin{aligned}
\Phi(x_i) \le V_L(x_i) \le \Psi(x_i), \; \forall x_i \in \bar{\Omega}^N \cap \Omega^-, \\
\Phi(x_i) \le V_R(x_i) \le \Psi(x_i), \; \forall x_i \in \bar{\Omega}^N \cap \Omega^+.
\end{aligned}$$

From (2.4) $u_\varepsilon(0) \le v_\varepsilon(d^-) \le 0$ and $u_\varepsilon(1) \ge v_\varepsilon(d^+) \ge 0$, we can use the following mesh function:

$$A_1(x_i) = u_\varepsilon(0),\ x_i \le d, \quad A_1(x_i) = \frac{v_\varepsilon(1)(x_i - d)}{1-d},\ x_i \ge d,$$
$$B_1(x_i) = v_\varepsilon(0)(1 - \frac{x_i}{d}),\ x_i \le d, \quad B_1(x_i) = u_\varepsilon(1),\ x_i \ge d,$$

to show that V_L and V_R exist using the previous theorem. Hence $A_1 \le V_L \le B_1 \le 0,\ 0 \le A_1 \le V_R \le B_1$ and thus $b_1(V_L) \le -1,\ b_2(V_R) \ge 1$.

Lemma 5.4. *Assume* (3.5)*. Given any V_L, V_R, solutions to* (5.3)*, we have that*

$$\begin{aligned} |V_L(x_i) - v_\varepsilon(x_i)| &\le CN^{-1}x_i,\ x_i \in \bar\Omega_\varepsilon^N \cap \Omega^-, \\ |V_R(x_i) - v_\varepsilon(x_i)| &\le CN^{-1}(1-x_i),\ x_i \in \bar\Omega_\varepsilon^N \cap \Omega^+, \end{aligned}$$

where v_ε is the unique regular component defined in (2.8).

Proof. We outline the proof for the first inequality. An analogous argument will establish the second inequality. For all $x_i \in \Omega_\varepsilon^N \cap \Omega^-$,

$$\begin{aligned} \left\{\varepsilon\delta^2 + b_1(V_L)D^-\right\}(V_L - v_\varepsilon) &= \varepsilon v_\varepsilon'' + b_1(v_\varepsilon)v_\varepsilon' - \left\{\varepsilon\delta^2 + b_1(V_L)D^-\right\}(v_\varepsilon) \\ &= \varepsilon(v_\varepsilon'' - \delta^2 v_\varepsilon) + b_1(V_L)(v_\varepsilon' - D^- v_\varepsilon) + (b_1(v_\varepsilon) - b_1(V_L))v_\varepsilon' \\ &= \varepsilon(v_\varepsilon'' - \delta^2 v_\varepsilon) + b_1(V_L)(v_\varepsilon' - D^- v_\varepsilon) + c(v_\varepsilon - V_L)v_\varepsilon'\ . \end{aligned}$$

Introduce the linear difference operator

$$M_V^N Z := \left(\varepsilon\delta^2 + b_1(V_L)D^- + cv_\varepsilon'\right)Z.$$

Note that $\|cv_\varepsilon'\| \le C$ and so, by Lemma 8.3 in the appendix, this finite difference operator satisfies a discrete comparison principle, provided that ε is sufficiently small such that

$$b_1^2(V_L) - 4\varepsilon cv_\varepsilon' > 0, \quad \forall x_i \in \Omega^N \cap\ \Omega^-. \tag{5.4}$$

Using the bounds in Lemma 4.2 and standard local truncation error estimates, we get

$$\left|M_V^N(V_L - v_\varepsilon)(x_i)\right| \le CN^{-1}.$$

With the two functions $\psi^\pm(x_i) = CN^{-1}x_i \pm (V_L - v_\varepsilon)(x_i)$, and the discrete comparison principle the proof is completed in the usual way. $\square$

To establish uniqueness for the discrete regular component V_L, we first obtain bounds on the discrete derivative of any possible regular component V_L.

Lemma 5.5. *Assume* (3.5)*. For any V_L, we have the following ε-uniform bounds*

$$|D^-V_L(x_i)| \le C,\ x_i \le d - \sigma_1 \quad \text{and} \quad |D^-V_L(x_i)| \le C\left(1 + \frac{N^{-1}}{\varepsilon}\right),\ d - \sigma_1 < x_i \le d.$$

Proof. Note that $D^-V_L(x_i) = D^-(V_L - v_\varepsilon)(x_i) + D^-v_\varepsilon(x_i) - v_\varepsilon'(x_i) + v_\varepsilon'(x_i)$. We also have $\|v_\varepsilon'\| \le C$ and, as in [4, page 60], $|D^-v_\varepsilon(x_i) - v_\varepsilon'(x_i)| \le CN^{-1}$. Hence,

$$|D^-V_L(x_i)| \le |D^-(V_L - v_\varepsilon)(x_i)| + C.$$

On $(0, d-\sigma_1]$, using the previous bound on $|(V_L - v_\varepsilon)(x_i)|$, we get that $|D^-(V_L - v_\varepsilon)(x_i)| \le C$. As in [4, pp. 61 and 62], $\varepsilon|D^-(V_L - v_\varepsilon)(x_i)| \le CN^{-1}$ on $(d-\sigma_1, d]$, where we note that we use

$$|b_1(V_L)(x_i) - b_1(V_L)(x_{i-1})| = c|V_L(x_i) - V_L(x_{i-1})|$$
$$\le |V_L(x_i) - v_\varepsilon(x_i)| + |v_\varepsilon(x_i) - v_\varepsilon(x_{i-1})| + |V_L(x_{i-1}) - v_\varepsilon(x_{i-1})| \le CN^{-1}.$$

This completes the proof. □

Lemma 5.6. *There exist unique solutions V_L and V_R to the discrete problems* (5.3) *and* (3.5).

Proof. Assume the contrary. Let V_L^+, V_L^- be two mesh solutions, then

$$\varepsilon\delta^2 V_L^- + b_1(V_L^-)D^-V_L^- = \varepsilon\delta^2 V_L^+ + b_1(V_L^+)D^-V_L^+, \ x_i < d,$$
$$(V_L^+ - V_L^-)(0) = (V_L^+ - V_L^-)(d) = 0.$$

Thus $\varepsilon\delta^2(V_L^+ - V_L^-) + b_1(V_L^+)D^-(V_L^+ - V_L^-) + cD^-V_L^-(V_L^+ - V_L^-) = 0$. From the previous lemma and Lemma 8.3, the linear difference operator

$$L_\varepsilon^N Z := \varepsilon\delta^2 Z + b_1(V_L^+)D^-Z + (cD^-V_L^-)Z$$

satisfies a discrete comparison principle. This guarantees uniqueness. □

We are now ready to state the discrete counterpart to Theorem 3.3. First we define the discrete barrier functions $\Xi_L(x_i;\gamma), \Xi_R(x_i;\gamma)$ as the solutions of

$$\varepsilon\delta^2\Xi_L + (-1 + c\gamma)D^-\Xi_L = -\delta_1, \ x_i \in (0,d), \quad \Xi_L(0) = u_\varepsilon(0), \ \Xi_L(d) = \gamma,$$
$$\varepsilon\delta^2\Xi_R + (1 + c\gamma)D^+\Xi_R = \delta_2, \ x_i \in (d,1), \quad \Xi_R(d) = \gamma, \ \Xi_R(1) = u_\varepsilon(1).$$

Theorem 5.7. *There exists a solution U_ε^N to the discrete problem* (5.2), (3.5) *and*

$$\begin{aligned} V_L(x_i) &\le U_\varepsilon^N(x_i) \le \Xi_L(x_i; v_R(d)), \quad x_i \le d, \\ \Xi_R(x_i; v_L(d)) &\le U_\varepsilon^N(x_i) \le V_R(x_i), \quad x_i \ge d. \end{aligned}$$

Proof. The argument is the discrete analogue of the argument given to establish the existence of the continuous solution. Define $U_L(x_i;\gamma), U_R(x_i;\gamma)$ to be the solutions of the problems

$$\varepsilon\delta^2 U_L + b_1(U_L)D^-U_L = -\delta_1, \ x_i \in (0,d), \quad U_L(0) = u_\varepsilon(0), \ U_L(d) = \gamma,$$
$$\varepsilon\delta^2 U_R + b_2(U_R)D^+U_R = \delta_2, \ x_i \in (d,1), \quad U_R(d) = \gamma, \ U_R(1) = u_\varepsilon(1).$$

From assumption (3.5), we have that for all $\gamma \in [v_L(d), v_R(d)]$ both problems have a solution $U_L(x_i;\gamma), U_R(x_i;\gamma)$ and

$$V_L(x_i) \le U_L(x_i;\gamma) \le \Xi_L(x_i;\gamma), \quad \Xi_R(x_i;\gamma) \le U_R(x_i;\gamma) \le V_R(x_i).$$

Note the following:

$$\begin{aligned} \varepsilon D^-V_L(d) &= \varepsilon v_L'(d^-) + \varepsilon(D^-V_L(d) - v_L'(d)) = O(\varepsilon) + O(N^{-1}), \\ \varepsilon D^+\Xi_R(d;\gamma) &= \varepsilon\xi_R'(d^+;\gamma) + \varepsilon(D^+\Xi_R(d;\gamma) - \xi_R'(d^+;\gamma)) \\ &\ge \frac{(1+c\gamma)(u_\varepsilon(1) - \gamma) - \delta_2(1-d)}{2} + O(\varepsilon) + O(N^{-1}). \end{aligned}$$

Hence, for ε sufficiently small and N sufficiently large,

$$D^+V_R(d) \le D^-\Xi_L(d; v_R(d)) \quad \text{and} \quad D^+\Xi_R(d; v_L(d)) \ge D^-V_L(d).$$

Use the following:

$$\Phi = \begin{cases} V_L(x_i), & \text{if } x_i \le d, \\ \Xi_R(x_i; v_L(d)), & \text{if } x_i \ge d, \end{cases} \qquad \Psi = \begin{cases} \Xi_L(x_i; v_R(d)), & \text{if } x_i \le d, \\ V_R(x_i), & \text{if } x_i \ge d, \end{cases}$$

as lower and upper mesh solutions to establish the existence of U_ε. □

Remark 5.8. If there exists a solution U_ε to the discrete problem (5.2), (3.5) with the additional property that

$$\begin{aligned} &(5.5\text{a}) \qquad |DU(x_i)| \le C\Big(1+\frac{N^{-1}}{\varepsilon}\Big), \quad d-\sigma_1 < x_i < d+\sigma_2, \\ &(5.5\text{b}) \qquad |DU(x_i)| \le C, \quad \text{otherwise,} \end{aligned}$$

then this solution is unique. This follows by observing that if there are two solutions U_1 and U_2 satisfying (5.5), then

$$\varepsilon\delta^2(U_2-U_1) + b(x,U_2)D(U_2-U_1) + cDU_1(U_2-U_1) = 0,$$

and so by Lemma 8.4, $(U_2 - U_1) = 0$.

Given any discrete solution U of (5.2), (3.5) we can define W_L and W_R using

$$W_L = U - V_L,\ x_i \le d, \quad W_R = U - V_R,\ x_i \ge d.$$

These functions $W_L : \overline{\Omega}_\varepsilon^N \cap [0,d] \to \mathbb{R}$ and $W_R : \overline{\Omega}_\varepsilon^N \cap [d,1] \to \mathbb{R}$ exist, are uniformly bounded, and satisfy the following system of finite difference equations:

$$\begin{aligned} &(5.6\text{a}) \quad (\varepsilon\delta^2 + (b_1(W_L) + cV_L)D^- + cD^-V_L)(W_L) = 0,\ x_i \in \Omega_\varepsilon^N \cap \Omega^-, \\ &(5.6\text{b}) \quad (\varepsilon\delta^2 + (b_2(W_R) + cV_R)D^+ + cD^+V_R)(W_R) = 0,\ x_i \in \Omega_\varepsilon^N \cap \Omega^+, \\ &(5.6\text{c}) \qquad W_L(0) = 0, \quad W_R(1) = 0, \\ &(5.6\text{d}) \qquad W_R(d) + V_R(d) = W_L(d) + V_L(d), \\ &(5.6\text{e}) \qquad D^+W_R(d) + D^+V_R(d) = D^-W_L(d) + D^-V_L(d). \end{aligned}$$

6. Error analysis

In the next theorem, we show that the discrete layer functions are small (in a discrete sense) exterior to the interior layer region.

Theorem 6.1. *When $\sigma_1 = \frac{2\varepsilon}{\theta_1}\ln N$ and $\sigma_2 = \frac{2\varepsilon}{\theta_2}\ln N$, we have that*

$$|W_L(x_i)| \le CN^{-1}, \quad x_i \le d-\sigma_1; \quad |W_R(x_i)| \le CN^{-1}, \quad x_i \ge d+\sigma_2,$$

where W_L and W_R are the solutions of the problems defined in (5.6).

Proof. Consider the case of $x_i \le d$. Let

$$B(x_i) = \Pi_{j=1}^{i}(1+\frac{\theta_1 h_j}{2\varepsilon})^j, \quad h_i = x_i - x_{i-1}.$$

Then

$$D^+B(x_i) = \frac{\theta_1}{2\varepsilon}B(x_i), \quad \left(1+\frac{\theta_1 h_i}{2\varepsilon}\right)D^-B(x_i) = \frac{\theta_1}{2\varepsilon}B(x_i),$$

and

$$\left(1+\frac{\theta_1 h_i}{2\varepsilon}\right)\delta^2 B(x_i) = \frac{\theta_1^2}{4\varepsilon^2}\left(2-\frac{h_{i+1}}{\bar{h}_i}\right)B(x_i).$$

Hence

$$\begin{aligned}&\left(1+\tfrac{\theta_1 h_i}{2\varepsilon}\right)\left(\varepsilon\delta^2+b_1(U)D^-+cD^-V_L\right)B(x_i)\\&\quad<\ \tfrac{\theta_1}{2\varepsilon}\left(\theta_1+b_1(U)+\tfrac{2\varepsilon}{\theta_1}(cD^-V_L)(1+\tfrac{\theta_1 h_i}{2\varepsilon})\right)B(x_i)<0.\end{aligned}$$

Then

$$\|W(x_i)\|\le C\|W(d)\|\frac{B(x_i)}{B(d)},\quad x_i\le d.$$

Thus, for $x_i\le d-\sigma_1$,

$$\|W(x_i)\|\le C\left(1+\frac{\theta_1 h}{2\varepsilon}\right)^{-N/4},\quad h=\frac{4\sigma_1}{N}.$$

Then, if $\sigma_1=\frac{2\varepsilon}{\theta_1}\ln N$, we have that $|W_L|\le CN^{-1}$, $x_i\le d-\sigma_1$ $|W_R|\le CN^{-1}$, $x_i\ge d+\sigma_2$. □

Thus, when $\sigma_1=\frac{2\varepsilon}{\theta_1}\ln N$, we have that

$$\begin{aligned}\text{(6.1a)}\quad |W_L(x_i)-w_\varepsilon(x_i)| &\le |W_L(x_i)|+|w_\varepsilon(x_i)|\le CN^{-1}+Ce^{-\theta_1\sigma_1/\varepsilon}\\ &\le CN^{-1},\quad x_i\le d-\sigma_1.\end{aligned}$$

Similarly, for $\sigma_2=\frac{2\varepsilon}{\theta_2}\ln N$, we obtain

$$\text{(6.1b)}\qquad |W_R(x_i)-w_\varepsilon(x_i)|\le CN^{-1},\quad x_i\ge d+\sigma_2.$$

These bounds and the bounds given in Lemma 5.4 together imply that the numerical approximations are essentially first order convergent at the mesh points outside the interior layer region $(d-\sigma_2,d+\sigma_2)$. To obtain an error estimate at the mesh points in the interior layer region, we assume the following implicit restriction on the data.

Assumption 2. Assume that the problem data for problem (1.1) are such that

$$\text{(6.2)}\qquad b^2(x_i,U_\varepsilon(x_i))-4\varepsilon c u_\varepsilon'(x_i)>0,\quad x_i\ne d.$$

As in Remark 3.4, from the bounds in Theorem 5.7, we have the strict inequality $|b(x_i,U_\varepsilon)|>\theta>0$. Hence, assumption (6.2) can be satisfied for certain problem data.

Theorem 6.2. *Assume that N is sufficiently large and ε is sufficiently small, independently of each other. Assume further that* (3.5) *and* (6.2) *hold. The continuous solution u_ε of problem* (1.1) *and any set of discrete solutions U_ε of* (5.2) *satisfy the following asymptotic error bound*

$$\|U_\varepsilon-u_\varepsilon\|_{\overline{\Omega}_\varepsilon^N}\le CN^{-1}(\ln N)^2,$$

where C is a constant independent of N and ε.

Proof. Consider first the case of $\sigma_1=\frac{2\varepsilon}{\theta_1}\ln N$ and $\sigma_2=\frac{2\varepsilon}{\theta_2}\ln N$. By Lemma 5.4 and (6.1), the result is valid for mesh points outside the interior layer region $(d-\sigma_1,d+\sigma_2)$. Hence

$$\text{(6.3)}\quad |U_\varepsilon(d-\sigma_1)-u_\varepsilon(d-\sigma_1)|\le CN^{-1}\quad\text{and}\quad |U_\varepsilon(d+\sigma_2)-u_\varepsilon(d+\sigma_2)|\le CN^{-1}.$$

On the other hand, in the layer region $(d-\sigma_1, d) \cup (d, d+\sigma_2)$, we have that

$$\begin{aligned}\varepsilon\delta^2(u_\varepsilon - U_\varepsilon) &+ b(x_i, U_\varepsilon)D(u_\varepsilon - U_\varepsilon) \\ &= \varepsilon(\delta^2 u_\varepsilon - u_\varepsilon'') + b(x_i, U_\varepsilon)Du_\varepsilon - b(x_i, u_\varepsilon)u_\varepsilon' \\ &= \varepsilon(\delta^2 u_\varepsilon - u_\varepsilon'') + \big(b(x_i, U_\varepsilon) - b(x_i, u_\varepsilon)\big)u_\varepsilon' + b(x_i, U_\varepsilon)\big(Du_\varepsilon - u_\varepsilon'\big) \\ &= \varepsilon(\delta^2 u_\varepsilon - u_\varepsilon'') + \big(c(U_\varepsilon - u_\varepsilon)\big)u_\varepsilon' + b(x_i, U_\varepsilon)\big(Du_\varepsilon - u_\varepsilon'\big).\end{aligned}$$

We introduce the linear difference operator

$$\begin{aligned}M_U^N Z &:= \big(\varepsilon\delta^2 + b(x_i, U_\varepsilon)D + cu_\varepsilon'\big)Z, \quad x_i \neq d, \\ M_U^N Z(d) &:= (D^+Z - D^-Z)(d).\end{aligned}$$

At the mesh point $x_i = d$, $[u_\varepsilon'] = [DU_\varepsilon] = 0$, and so

$$\begin{aligned}|(D^+ - D^-)(U_\varepsilon - u_\varepsilon)| &= |(D^- - D^+)(u_\varepsilon) + [u_\varepsilon']| \\ &\leq |u_\varepsilon'(d) - D^+u_\varepsilon(d)| + |u_\varepsilon'(d) - D^-u_\varepsilon(d)| \\ &\leq Ch\|u_\varepsilon''\|_{(x_{i-1}, x_{i+1})},\end{aligned}$$

where $h = \frac{4\sigma}{N}$ and $\sigma = \max\{\sigma_1, \sigma_2\}$ is the fine mesh size. Hence, using $u_\varepsilon = v_\varepsilon + w_\varepsilon$ and the bounds in Lemma 4.2,

$$\begin{aligned}M_U^N(u - U) &= \varepsilon(\delta^2 u_\varepsilon - u_\varepsilon'') + b(x_i, U_\varepsilon)\big(Du_\varepsilon - u_\varepsilon'\big), \quad x_i \neq d, \\ |M_U^N(u - U)(x_i)| &\leq Ch\left(1 + \frac{1}{\varepsilon^2}e^{-\frac{4\ln N}{N}|N/2 - i|}\right), \quad x_i \in (d - \sigma_1, d + \sigma_2).\end{aligned}$$

Note that

$$e^{-\frac{4\ln N}{N}} \leq \left(1 + \frac{4\ln N}{N}\right)^{-1}$$

and hence, we have a truncation error bound of the form

$$|M_U^N(u - U)(x_i)| \leq Ch(1 + \frac{1}{\varepsilon^2}(1 + \frac{4\ln N}{N})^{-|N/2 - i|}), \quad x_i \in (d - \sigma_1, d + \sigma_2).$$

The finite difference operator M_U^N satisfies a discrete comparison principle (see Lemma 8.4 in the appendix), provided that (6.2) is assumed. Consider the discrete barrier function

$$\Psi = -CN^{-1} - C\frac{N^{-1}\sigma^2}{\varepsilon^2}\begin{cases} \frac{x_i - (d - \sigma_1)}{\sigma_1}, & x_i \in \Omega_\varepsilon^N \cap (d - \sigma_1, d], \\ \frac{(d + \sigma_2) - x_i}{\sigma_2}, & x_i \in \Omega_\varepsilon^N \cap (d, d + \sigma_2). \end{cases}$$

Form the product $\lambda(x_i)\Psi(x_i)$, where $|b(U)| > \theta = \min\{\theta_1, \theta_2\}$, and we define

$$\lambda(x_i) = \begin{cases} (1 + \frac{\theta_1 h_1}{2\varepsilon})^{i - N/2}, & x_i \in \Omega_\varepsilon^N \cap (d - \sigma_1, d), \\ (1 + \frac{\theta_2 h_2}{2\varepsilon})^{N/2 - i} & x_i \in \Omega_\varepsilon^N \cap (d, d + \sigma_2). \end{cases}$$

Then

$$M_U^N(\lambda\Psi)(x_i) = \begin{cases} \varepsilon\lambda(x_{i+1})\delta^2\Psi + \tilde{a}\lambda(x_{i-1})D^-\Psi + \tilde{b}\Psi, & x_i \in \Omega_\varepsilon^N \cap (d - \sigma_1, d), \\ \varepsilon\lambda(x_{i-1})\delta^2\Psi + \hat{a}\lambda(x_{i+1})D^+\Psi + \hat{b}\Psi, & x_i \in \Omega_\varepsilon^N \cap (d, d + \sigma_2) \end{cases}$$

where, for N sufficiently large, and using the strict inequality $|b(x_i,U)| > \theta$,

$$\begin{aligned}
\tilde{a} &= b_1(U) + \theta_1 + \frac{\theta_1^2 h_1}{4\varepsilon} \\
&< b(U) + \theta_1 + CN^{-1}\ln N < 0, \quad x_i \in \Omega_\varepsilon^N \cap (d-\sigma_1, d), \\
\hat{a} &= b_2(U) - \theta_2 - \frac{\theta_2^2 h_2}{4\varepsilon} > 0, \quad x_i \in \Omega_\varepsilon^N \cap (d, d+\sigma_2), \\
\frac{\tilde{b}}{\lambda(x_{i-1})} &= \frac{\theta_1^2}{4\varepsilon} + \frac{\theta_1 b_1(U)}{2\varepsilon} + cu_\varepsilon'\left(1 + \frac{\theta_1 h_1}{2\varepsilon}\right) \\
&< \frac{1}{\varepsilon}\left(c\varepsilon u_\varepsilon' - \frac{\theta_1^2}{4} + CN^{-1}\ln N\right) < 0, \quad x_i \in \Omega_\varepsilon^N \cap (d-\sigma_1, d), \\
\frac{\hat{b}}{\lambda(x_{i+1})} &= \frac{\theta_2^2}{4\varepsilon} - \frac{\theta_2 b_2(U)}{2\varepsilon} + cu_\varepsilon'\left(1 + \frac{\theta_2 h}{2\varepsilon}\right) \\
&< \frac{1}{\varepsilon}\left(c\varepsilon u_\varepsilon' - \frac{\theta_2^2}{4} + CN^{-1}\ln N\right) < 0, \quad x_i \in \Omega_\varepsilon^N \cap (d, d+\sigma_2).
\end{aligned}$$

Also,

$$\varepsilon^2 D^+\Psi(x_i) = CN^{-1}\frac{\sigma^2}{\sigma_2},\ x_i > d, \quad -\varepsilon^2 D^-\Psi(x_i) = CN^{-1}\frac{\sigma^2}{\sigma_1},\ x_i < d,$$

and $\delta^2\Psi(x_i) = 0, x_i \neq d$. Hence, for $x_i \neq d$,

$$M_U^N(\lambda\Psi) \geq C|\tilde{a}|\left(1 + \frac{\theta h}{2\varepsilon}\right)^{-1}\lambda(x_i)\frac{N^{-1}\sigma}{\varepsilon^2} \geq |M_U^N(u-U)|.$$

Noting that $\theta_1 h_1 = \theta_2 h_2$, we have the following bound at the point of discontinuity $x_i = d$,

$$M_U^N(\lambda\Psi) = \lambda(d-h)\left(D^+\Psi(d) - D^-\Psi(d) - \frac{\theta_1+\theta_2}{2\varepsilon}\Psi(d)\right) \geq C\frac{h}{\varepsilon^2} \geq |M_U^N(u-U)(d)|.$$

Applying the discrete comparison principle to $\Psi \pm (U_\varepsilon - u_\varepsilon)$ over the interval $[d-\sigma_1, d+\sigma_2]$, we get

$$|U_\varepsilon(x_i) - u_\varepsilon(x_i)| \leq CN^{-1} + C\frac{N^{-1}\sigma^2}{\varepsilon^2} \leq CN^{-1}(\ln N)^2.$$

We complete the proof by considering the case where at least one of the two transition points σ_1, σ_2 takes the value $\frac{d}{2}$ or $\frac{1-d}{2}$. In all such cases $\varepsilon^{-1} \leq C\ln N$. Apply the above argument across the entire domain Ω^N to complete the proof. $\square$

Remark 6.3. Note that (6.2) is a restriction on the problem class. In this remark, we show that this restriction is satisfied if ε is sufficiently small and the data is further restricted. We first examine the restrictions placed on the data when we require that

$$b^2(x, u_\varepsilon) - 4\varepsilon c u_\varepsilon' > 0, \quad \forall x \in (0,1).$$

Note that $u_\varepsilon = v_\varepsilon + w_\varepsilon$ and so for ε sufficently small, $u_\varepsilon' > 0,\ x \in (0,1);\ u_\varepsilon''(x) > 0, x < d,\ u_\varepsilon''(x) < 0, x > d$. Thus

$$(b^2(x,u_\varepsilon) - 4\varepsilon c u_\varepsilon')(x) \geq \max\{(1 - cu_\varepsilon(d))^2, (1 + cu_\varepsilon(d))^2\} - 4\varepsilon c u_\varepsilon'(d),\ \forall x \in (0,1).$$

Note also that

$$\begin{aligned}\varepsilon u'(d) - \varepsilon u'_\varepsilon(0) &= \int_0^d (-\delta_1 + (1 - cu_\varepsilon(t))u_\varepsilon(t))\, dt \\ &= -d\delta_1 + \frac{1}{2c}\left(2cu_\varepsilon(d) - c^2u_\varepsilon^2(d) - 2cu_\varepsilon(0) + c^2u_\varepsilon^2(0)\right) \\ &= -d\delta_1 + \frac{1}{2c}\left((1 - cu_\varepsilon(0))^2 - (1 - cu_\varepsilon(d))^2\right),\end{aligned}$$

and so

$$(1 - cu_\varepsilon(d))^2 - 4\varepsilon cu'_\varepsilon(d) > 4c\delta_1 d + 3(1 - cu_\varepsilon(d))^2 - 2(1 - cu_\varepsilon(0))^2.$$

If $-\eta < cu_\varepsilon(0) < cu_\varepsilon(1) < \eta$, then $1 - \eta < 1 - cu_\varepsilon(1) < 1 - cu_\varepsilon(d) < 1 - cu_\varepsilon(0) < 1 + \eta$. This means that $(1 - cu_\varepsilon(d))^2 - 4\varepsilon cu'_\varepsilon(d) > 4c\delta_1 d + (1 - \eta)^2 - 8\eta = 4c\delta_1 d + 1 - 10\eta + \eta^2$. Hence, we require the data to be such that

$$4c \max\{\delta_1 d, \delta_2(1 - d)\} + 1 - 10\eta + \eta^2 > 0,$$

where $-\eta < cu_\varepsilon(0) < cu_\varepsilon(1) < \eta$ and $\delta_1 d < -u_\varepsilon(0)$ and $\delta_2(1 - d) < u_\varepsilon(1)$. For example, if $\eta = 0.1, c = 1$, $\delta_1 d < -u_\varepsilon(0) < 0.1$ and $\delta_2(1 - d) < u_\varepsilon(1) < 0.1$, then the data constraints (3.5) and (6.2) in Theorem 6.2 are both satisfied.

7. Numerical results

To solve the nonlinear difference scheme (5.2) we use the continuation method described in [8]. Table 1 displays the computed rates of convergence p_ε^N and the uniform rates of convergence p^N, using the double mesh principle (see [4] for details on how these quantities are calculated), when the numerical method (5.2) is applied to the problem (1.1) with $u(0) = -0.5, \delta_1 = 0.8, u(1) = 0.7, \delta_2 = 1.2, d = 0.5, c = 1$. Note that the conditions in (3.5) are satisfied for this data. The computed rates of convergence are in line with the theoretical rates of convergence established in Theorem 6.2.

TABLE 1. Table of computed orders p_ε^N and computed ε-uniform orders p^N for the numerical method (5.2) applied to problem (1.1) with $u(0) = -0.5, \delta_1 = 0.8, u(1) = 0.7, \delta_2 = 1.2, d = 0.5, c = 1$.

	Number of Intervals N					
ε	32	64	128	256	512	1024
2^{-1}	0.96	0.98	0.99	0.99	1.00	1.00
2^{-2}	0.92	0.96	0.98	0.99	0.99	1.00
2^{-3}	0.88	0.94	0.96	0.98	0.99	0.99
2^{-4}	0.77	0.86	0.93	0.96	0.98	0.99
2^{-5}	0.62	0.78	0.86	0.93	0.96	0.98
2^{-6}	-0.25	0.63	0.78	0.86	0.93	0.96
2^{-8}	0.24	0.28	0.48	0.58	0.75	0.77
2^{-10}	0.26	0.29	0.48	0.59	0.76	0.77
2^{-12}	0.27	0.29	0.48	0.59	0.76	0.77
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-23}	0.27	0.29	0.48	0.59	0.76	0.77
p^N	0.29	0.63	0.63	0.59	0.76	0.75

8. Appendix on discrete comparison principles

Consider the following linear problem:

$$L_1u := \varepsilon u'' + pu' + qu = f,\ x \in (0,1), \quad u(0) = u_0,\ u(1) = u_1,$$
$$p \ge \alpha > 0,\ q \le \beta, \quad \alpha^2 - 4\varepsilon\beta > 0.$$

Lemma 8.1. *If $w(0) \le 0,\ w(1) \le 0,\ L_1w \ge 0$, then $w \le 0, x \in [0,1]$.*

Proof. If $\beta \le 0$, then the standard proof by contradiction argument applies. For $\beta > 0$, use the transformation $w = e^{-\frac{\alpha}{2\varepsilon}x}v$. □

Consider the corresponding discrete problem on an arbitrary mesh Ω^N,

$$L_1^N U := \varepsilon\delta^2 U + pD^+U + qU = f,\ x_i \in \Omega^N, \quad U(0) = u_0,\ U(1) = u_1, \quad p \ge \alpha > 0.$$

If $q \le 0$, then the standard discrete comparison principle holds for L_1^N. Below we extend this comparison principle to the case of $q > 0$. As in the proof of the continuous operator, consider the transformation

$$W(x_i) = \lambda(x_i)V(x_i),$$

where λ will be specified below. For any such mesh function λ,

$$\begin{aligned} D^+(\lambda(x_i)V(x_i)) &= \lambda(x_{i+1})D^+V(x_i) + V(x_i)D^+\lambda(x_i), \\ D^-(\lambda(x_i)V(x_i)) &= \lambda(x_{i-1})D^-V(x_i) + V(x_i)D^-\lambda(x_i), \\ \delta^2(\lambda(x_i)V(x_i)) &= V(x_i)\delta^2(\lambda(x_i)) + \frac{1}{\bar{h}_i}\left(\lambda(x_{i+1})D^+V(x_i) - \lambda(x_{i-1})D^-V(x_i)\right), \end{aligned}$$

where $\bar{h}_i = (h_i + h_{i+1})/2$. Hence

$$\delta^2(\lambda(x_i)V(x_i)) = V(x_i)\delta^2(\lambda(x_i)) + \frac{(\lambda(x_{i+1}) - \lambda(x_{i-1}))}{\bar{h}_i}D^+V(x_i) + \lambda(x_{i-1})\delta^2(V(x_i)).$$

Then, for $W(x_i) = \lambda(x_i)V(x_i)$ we have

$$(\varepsilon\delta^2W + pD^+W + qW)(x_i) = \left[\varepsilon\lambda(x_{i-1})\delta^2V + \lambda(x_{i+1})\hat{p}D^+V + \hat{q}V\right](x_i)\ ,$$

where

$$\hat{p} = \frac{\varepsilon}{\bar{h}_i}\left(\left(1 + \frac{p\bar{h}_i}{\varepsilon}\right) - \frac{\lambda(x_{i-1})}{\lambda(x_{i+1})} <\right),\ \hat{q} = \varepsilon\delta^2(\lambda(x_i)) + pD^+(\lambda(x_i)) + q\lambda(x_i)\ .$$

In the following three lemmas, we assume that $h_i \le CN^{-1}\ln N$ and that ε is sufficiently small (independently of N) and N is sufficiently large (independently of ε).

Lemma 8.2. *Assume that $p \ge \alpha > 0$. Under any one of the following three assumptions:*

(1) $q(x_i) \le C_2, \forall x_i$,
(2) $p > \alpha > 0$, $\alpha^2 - 4\varepsilon q > 0$ *and* Ω^N *is a uniform mesh with* $h/\varepsilon \le CN^{-1}\ln N$,
(3) $\Omega^N = \Omega_\varepsilon^N$ *is a piecewise uniform mesh which uses a uniform mesh in each of the subintervals* $[0,\sigma$ *and* $[\sigma,1]$ *(with a fine mesh step* $h \le C\varepsilon N^{-1}\ln N$ *and a coarse mesh step* $H \le CN^{-1}\ln N$*) and*

$$q(x_i) \le C_1(1 + (\varepsilon N)^{-1}), x_i < \sigma, \qquad q(x_i) \le C_2, x_i \ge \sigma;$$

then for any mesh function W, if $W(0) \le 0,\ W(1) \le 0,\ L_1^N W \ge 0$, then $W(x_i) \le 0,\ \forall x_i \in \Omega^N$.

Proof. We employ functions of the form $\lambda(x_i) = \Pi_{j=1}^{i}(1+\theta_j h_j)^{-1}$, which satisfy

$$D^+\lambda(x_i) = -\theta_{i+1}\lambda(x_{i+1}), \quad D^-\lambda(x_i) = -\theta_i\lambda(x_i),$$

$$D^+\lambda(x_i) - D^-\lambda(x_i) = ((\theta_i - \theta_{i+1}) + \theta_{i+1}\theta_i h_{i+1})\lambda(x_{i+1}),$$

$$1 + \frac{p\bar{h}_i}{\varepsilon} - \frac{\lambda(x_{i-1})}{\lambda(x_{i+1})} = \frac{\bar{h}_i}{\varepsilon}\left(p - \frac{\varepsilon}{\bar{h}_i}(h_i\theta_i + h_{i+1}\theta_{i+1} + h_i\theta_i h_{i+1}\theta_{i+1})\right),$$

$$\hat{q}(x_i) = \lambda(x_{i+1})\left[(1+\theta_{i+1}h_{i+1})\left(\frac{\varepsilon}{\bar{h}_i}\theta_i + q(x_i)\right) - \theta_{i+1}\left(\frac{\varepsilon}{\bar{h}_i} + p(x_i)\right)\right].$$

In each of the three cases, we choose the θ_j so that $\hat{p} \ge 0, \hat{q} < 0$ and then the normal proof by contradiction argument can be applied. Assume $W_i > 0$ and let $V_j = \max V_i > 0$, then $D^+V_j \le 0, D^-V_j \ge 0,\ \delta^2 V_j \le 0,\ L^N W_j \le 0$.

Case 1. Take, $\theta_j = \theta = \frac{2C_2+1}{\alpha}$. Then if ε is sufficiently small (independently of N) and N is sufficiently large (independently of ε),

$$1 + \frac{p\bar{h}_i}{\varepsilon} - \frac{\lambda(x_{i-1})}{\lambda(x_{i+1})} \ge \frac{\bar{h}_i}{\varepsilon}(\alpha - 2\varepsilon\theta(1+\theta h)) \ge 0, \quad h = \min\{h_i, h_{i+1}\},$$

$$\hat{q} \le \lambda(x_{i+1})\big(2\varepsilon\theta^2 + q - \alpha\theta + q\theta h_{i+1}\big) < 0.$$

Case 2. Take $\theta_j = \frac{\alpha}{2\varepsilon}$. Then, if N is sufficiently large (independently of ε),

$$1 + \frac{p\bar{h}_i}{\varepsilon} - \frac{\lambda(x_{i-1})}{\lambda(x_{i+1})} = \frac{h}{\varepsilon}\left(p - \alpha(1 + \frac{\alpha h}{2\varepsilon})\right) \ge 0,$$

using the strict inequality $p > \alpha$ and

$$\hat{q} \le \lambda(x_{i+1})\big(\varepsilon\theta^2 + q - \alpha\theta + q\theta h\big) < 0,$$

using the strict inequality $4\varepsilon q < \alpha^2$.

Case 3. If $\varepsilon N \ge 1$, then follow the argument in Case 1. Otherwise, take

$$\theta_i = \frac{\zeta_1}{\varepsilon N},\ i \le N/2, \quad \theta_i = \zeta_2, i > N/2,$$

where

$$\zeta_1 = \frac{2C_1+1}{\alpha}, \quad \alpha\zeta_2 > q + \zeta_1.$$

In the layer region, when $i < N/2$ and $h \le C\varepsilon N^{-1}\ln N$,

$$1 + \frac{p\bar{h}_i}{\varepsilon} - \frac{\lambda(x_{i-1})}{\lambda(x_{i+1})} \ge \frac{h}{\varepsilon}\left(p - \frac{\zeta_1}{N}(2 + \frac{\zeta_1 h}{N\varepsilon})\right) \ge 0, \text{ for any } \ \zeta_1,$$

$$\frac{\hat{q}}{\lambda(x_{i+1})} = \frac{1}{\varepsilon N}(q\varepsilon N - p\zeta_1) + \frac{\zeta_1}{N\varepsilon}(qh + \frac{\zeta_1}{N}) < 0,$$

using $q\varepsilon N \le C_1(\varepsilon N + 1) \le 2C_1$. Outside the layer region, when $i > N/2$, for ε sufficiently small,

$$1 + \frac{p\bar{h}_i}{\varepsilon} - \frac{\lambda(x_{i-1})}{\lambda(x_{i+1})} = \frac{H}{\varepsilon}(p - \varepsilon\zeta_2(2 + \zeta_2 H)) \ge 0$$

and

$$\frac{\hat{q}}{\lambda(x_{i+1})} \le q - \alpha\zeta_2 + \zeta_2(2\varepsilon\zeta_2 + qH) < 0.$$

At the transition point, when $i = N/2$ and N is sufficiently large,

$$1 + \frac{p\bar{h}_i}{\varepsilon} - \frac{\lambda(x_{i-1})}{\lambda(x_{i+1})} = \frac{\bar{h}}{\varepsilon}\left(p - (h\zeta_1 + \varepsilon NH\zeta_2 + h\zeta_1\zeta_2 H)\right) \geq 0,$$

$$\frac{\hat{q}(x_i)}{\lambda(x_{i+1})} = (1+\zeta_2 H)(q(x_i)+\zeta_1) - \zeta_2(\alpha + \varepsilon N) < 0,$$

using the strict inequality $\alpha\zeta_2 > q + \zeta_1$. □

In the case of a negative convective coefficient $p < 0$, we employ the operator

$$L_2^N U := \varepsilon\delta^2 U + pD^- U + qU.$$

Lemma 8.3. *Assume that $p \leq -\alpha < 0$ and any one of the following:*

(1) $q(x_i) \leq C_2, \forall x_i$,

(2) $p < -\alpha < 0$, $\alpha^2 - 4\varepsilon q > 0$ *and Ω^N is a uniform mesh with* $h/\varepsilon \leq CN^{-1}\ln N$,

(3) $\Omega^N = \Omega_\varepsilon^N$ *is a piecewise uniform mesh and*

$$q(x_i) \leq C_1(1+(\varepsilon N)^{-1}), x_i > 1-\sigma,\ q(x_i) \leq C_2, x_i \leq 1-\sigma;$$

then for any mesh function W, if $W(0) \leq 0$, $W(1) \leq 0$, $L_2^N W \geq 0$, then $W(x_i) \leq 0,\ \forall x_i \in \Omega^N$.

Proof. The proof is analogous to the case of $p > 0$. As before, if $W(x_i) = \lambda(x_i)V(x_i)$, then

$$(\varepsilon\delta^2 W + pD^- W + qW)(x_i) = \left[\varepsilon\lambda(x_{i+1})\delta^2 V + \tilde{p}\lambda(x_{i-1})D^- V + \tilde{q}V\right](x_i)\ ,$$

where

$$\tilde{p} = \frac{\varepsilon}{\bar{h}_i}\left(\frac{\lambda(x_{i+1})}{\lambda(x_{i-1})} - (1 - \frac{p\bar{h}_i}{\varepsilon})\right),\ \tilde{q} = \varepsilon\delta^2(\lambda(x_i)) + pD^-(\lambda(x_i)) + q\lambda(x_i)\ .$$

Consider functions of the form $\lambda(x_i) = \Pi_{j=1}^i(1+\theta_j h_j)$, which satisfy

$$\begin{aligned}
D^+\lambda(x_i) = \theta_{i+1}\lambda(x_i), \qquad & D^-\lambda(x_i) = \theta_i\lambda(x_{i-1}),\\
D^+\lambda(x_i) - D^-\lambda(x_i) &= ((\theta_{i+1}-\theta_i) + \theta_{i+1}\theta_i h_i)\lambda(x_{i-1}),\\
\frac{\lambda(x_{i+1})}{\lambda(x_{i-1})} - \left(1 - \frac{p\bar{h}_i}{\varepsilon}\right) &= \frac{\bar{h}_i}{\varepsilon}\left(p + \frac{\varepsilon}{\bar{h}_i}(h_i\theta_i + h_{i+1}\theta_{i+1} + h_i\theta_i h_{i+1}\theta_{i+1})\right),\\
\frac{1}{\lambda(x_{i-1})}\hat{q} &\leq (1+\theta_i h_i)\left(\frac{\varepsilon}{\bar{h}_i}\theta_{i+1} + q\right) - \theta_i\left(\frac{\varepsilon}{\bar{h}_i} + \alpha\right).
\end{aligned}$$

The proof is analogous to the previous proof. □

In the case of a discontinuous convective coefficient $p < -\alpha < 0, x < d$, and $p > \alpha > 0, x > d$ we use the upwind finite difference operator

$$L_3^N U := \varepsilon\delta^2 U + pDU + qU.$$

Lemma 8.4. *Under any one of the following assumptions:*

(1) $q(x_i) \leq C_2, \forall x_i$,

(2) $\alpha^2 - 4\varepsilon q > 0$ *and Ω^N is a uniform mesh with* $h/\varepsilon \leq CN^{-1}\ln N$,

(3) $\Omega^N = \Omega_\varepsilon^N$ *is a piecewise uniform mesh, and*

$$q(x_i) \leq C_1(1+(\varepsilon N)^{-1}), d-\sigma < x_i < d+\sigma,\ q(x_i) \leq C_2,\ \text{otherwise};$$

then for any mesh function W, *if* $W(0) \leq 0,\ W(1) \leq 0,\ L_3^N W \geq 0,\ D^+W(d) \geq D^-W(d)$, *we have* $W(x_i) \leq 0,\ \forall x_i \in \Omega^N$.

Proof. Use functions of the form:

$$\lambda(x_i) = \Pi_{j=1}^{i}(1+\theta_j h_j), i \leq N/2, \quad \lambda(x_i) = \lambda(d)\Pi_{j=i-N/2}^{i}(1+\theta_j h_j)^{-1}, i > N/2.$$

In all three cases, using the choices from the previous two lemmas, we get $\lambda(x_{N/2-1}) = \lambda(x_{N/2+1})$. Then

$$\delta^2 W(d) = \lambda(x_{N/2-1})\left(\delta^2 V(d) - 2\frac{\theta}{h}V(d)\right).$$

If $W > 0$ and $\max V = V(d) > 0$, then $\delta^2 W(d) < 0$, which is a contradiction. □

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Department of Computer Science, Kent State University, Kent, Ohio 44242
E-mail address: farrell@cs.kent.edu

School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland
E-mail address: eugene.oriordan@dcu.ie

Institute of Mathematics and Mechanics, Russian Academy of Sciences, Ekaterinburg, Russia
E-mail address: shishkin@imm.uran.ru