The Hat Problem: A Study in Probability
Michael George Billig

In his Elementary Probability and Statistics (MATH 10041) class, Dr. Bathi Kasturiarachi asked his students to create projects in which they would produce data, analyze this data, and, finally, make logical inferences from it. "The Hat Problem: A Study in Probability" is Michael George Billig's analysis of a hypothetical game show. He demonstrates how binary matrices can improve the odds of the show's players' abilities to guess the correct answer.

In all probability, nearly everyone wants to be a millionaire, or at least a winner. Game shows, horse races, the casinos in Vegas, and state lotteries all rely on probability and the ever-present hope of beating the odds. Be assured that the organizers of such activities are aware of the methods of increasing the odds in their favor, and have made the necessary adjustments. While they have not made winning impossible (such obvious exploitation and manipulation would be bad for business), they have made winning as unlikely an event as they can. Is there any way for the player to beat the odds?

Nature of the Study: Introduction and Statement of the Problem
At a mathematical game show with $n$ players, the host blindfolds the contestants and puts colored hats on their heads. The color of each person's hat—red or blue—is determined by a coin toss, independently of all the others. After the blindfolds come off, each player can see his teammates' hats, but not his own. When the host gives a signal, all players simultaneously either guess the color of their own hats or pass. If there are no incorrect guesses and at least one correct guess, the players share a $1,000,000$ prize. There is no communication between the players during the show, but they are told the rules in advance and are allowed to discuss their strategy. What should they do to maximize their chances of success?

This problem deals with developing a strategy by which a certain number of players ($n$) can increase their odds of winning a shared prize. The contestants are only able to discuss their strategy prior to hat color assignment. After this point, the only information they receive is the color outcomes of all other players, not their own, and an opportunity to guess their own hat color or pass.

The odds of one person guessing his own hat color with others passing is $1$ in $2$, or $\frac{1}{2}$ (50%). If all $n$ players guess, the odds of group success gets worse ($\frac{1}{2^n}$). In this case, the best probability is with all players passing except one who would guess his/her hat color. Obviously we want to increase our odds of success beyond those of one designated player guessing. How to increase the odds to better than 50%? A process using binary matrices allows us to improve those odds through a series of simple calculations and a response to the outcome of those calculations. For brevity and simplicity, we will use a smaller number of players and demonstrate the operation of the matrix as it is applied to each player's observations.

Methodology
For a population of $n = 2$, we can easily assign binary numbers 01 (vertically assigned as ac) and 10 (bd) for one and two respectively, and construct the first section of the matrix.

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
We then assign single digits to the two colors: blue = 0, and red = 1. After the blindfolds are removed, player A mentally assigns the upper register (x) a 0, and observing the color of player B's hat, assigns either a 0 or 1 to the lower register (y).

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} \begin{pmatrix}
x \\
y \\
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}
\]

The resulting numbers are then calculated according to the matrix multiplication diagram below and entered in the third set of parentheses.

\[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} \begin{pmatrix}
x \\
y \\
\end{pmatrix} = \begin{pmatrix}
ax + by \\
(cx + dy) \\
\end{pmatrix}
\]

Player A then makes a guess or passes, based on the outcome of the numbers in the third set of parentheses as follows: if all are 0s, guess red (1) for self, and if one's own binary number is in the third set of parentheses, guess blue (0) for self, and if there are any other sequences, then player A passes. Player B performs nearly the same operation, instead assigning the lower register (y) a 0 and in the upper register (x) places the color number (1 or 0) which corresponds to the observed color of A's hat.

Using this method, the outcomes and subsequent guesses or passes are calculated for n=2 (See Appendix 1.) and n=3 (See Appendix 2.).

**Analysis of Data**

From these cumulative guesses and passes we can compute the improved odds of success for each population in acquiring at least one correct guess and no wrong guesses.

<table>
<thead>
<tr>
<th>n</th>
<th>Odds using random, one-player guesses</th>
<th>Odds using binary matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 in 2 (0.50)</td>
<td>2 in 4</td>
</tr>
<tr>
<td>3</td>
<td>1 in 2 (0.50)</td>
<td>6 in 8</td>
</tr>
<tr>
<td>7</td>
<td>1 in 2 (0.50)</td>
<td>112 in 128</td>
</tr>
</tbody>
</table>

**Conclusion**

Further discovery and development of a spreadsheet program will allow for comparison of the odds increase for larger populations. It has been shown that this method works best with numbers of the form \(2^{n-1}\). Can we show somehow by manipulating numbers other than these to behave like 3, 7, 15, etc.? Further study using octal, decimal, or hexadecimal matrices, different bases, or other methods may prove fruitful.

Another possible strategy would be to artificially manipulate the number of players to equal one of the \(2^{n-1}\) numbers by disregarding the appropriate number of players (they would automatically pass every time) so that only 3 or 7 or 15, etc., players are calculated into the matrices in order to maximize the odds. However, it is important to remember that although higher n values mean greater odds of success, they also require a larger, more elaborate matrix, and all of the more elaborate operations which would follow.
From this hypothetical contest, we have seen that one is able to create a method to correlate unrelated known facts, to incorporate them into a mathematical operation, and to arrive at a conclusion which will increase the odds well beyond those of random guessing (here, up to 87.5%). Granted, this is only a hypothetical game show, but in the real world there exist situations in which similar mathematical operations can be applied in order to increase the possibility of success. Care to place a bet?

Acknowledgements and Resources


Private communications with Dr. Bathi Kasturiarachi.
Appendix 1
Three players A & B (n = 2)
Red Hat = 1, Blue Hat = 0

<table>
<thead>
<tr>
<th>Player</th>
<th>Trial #1</th>
<th>Trial #2</th>
<th>Trial #3</th>
<th>Trial #4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Binary identity # for player A = \[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \], for player B = \[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \].

Strategy for "Observation": Assign either 0 or 1 to other player by observing the hat color. Always assign 0 to self.

Strategy for "Decision": If outcome is all 0’s, tell game show host you guess 1 (red) for yourself. If outcome is your own binary number, tell game show host you guess 0 (blue) for yourself. All other cases you pass.

<table>
<thead>
<tr>
<th>Player</th>
<th>Observe</th>
<th>Outcome</th>
<th>Decision</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ 0 \end{pmatrix} ]</td>
<td>pass</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Lose</td>
</tr>
</tbody>
</table>

Trial #1: \[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

<table>
<thead>
<tr>
<th>Player</th>
<th>Observe</th>
<th>Outcome</th>
<th>Decision</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ 0 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ 0 \end{pmatrix} ]</td>
<td>pass</td>
</tr>
</tbody>
</table>

Trial #2: \[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

<table>
<thead>
<tr>
<th>Player</th>
<th>Observe</th>
<th>Outcome</th>
<th>Decision</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ 0 \end{pmatrix} ]</td>
<td>pass</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Win!</td>
</tr>
</tbody>
</table>

Trial #3: \[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

<table>
<thead>
<tr>
<th>Player</th>
<th>Observe</th>
<th>Outcome</th>
<th>Decision</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ 0 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 \ 0 \end{pmatrix} ]</td>
<td>red</td>
</tr>
</tbody>
</table>

Trial #4: \[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

<table>
<thead>
<tr>
<th>Player</th>
<th>Observe</th>
<th>Outcome</th>
<th>Decision</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 \ 0 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 \ 0 \end{pmatrix} ]</td>
<td>red</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Lose</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Player</th>
<th>Observe</th>
<th>Outcome</th>
<th>Decision</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>[ \begin{pmatrix} 0 \ 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 \ 0 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 \ 0 \end{pmatrix} ]</td>
<td>red</td>
</tr>
</tbody>
</table>
Appendix 2
Three players A, B, & C (n = 3)
Red Hat = 1, Blue Hat = 0

<table>
<thead>
<tr>
<th>Player</th>
<th>Trial #1</th>
<th>Trial #2</th>
<th>Trial #3</th>
<th>Trial #4</th>
<th>Trial #5</th>
<th>Trial #6</th>
<th>Trial #7</th>
<th>Trial #8</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Binary identity # for player A = \[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix},
\] for player B = \[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix},
\] and player C = \[
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}.
\]

Strategy for "Observation": Assign either 0 or 1 to other players by observing their hat color. Always assign 0 to self.
Strategy for "Decision": If outcome is all 0’s, tell game show host you guess 1 (red) for yourself. If outcome is your own binary number, tell game show host you guess 0 (blue) for yourself. All other cases you pass.

<table>
<thead>
<tr>
<th>Player</th>
<th>Observe</th>
<th>Outcome</th>
<th>Decision</th>
<th>Result</th>
</tr>
</thead>
</table>
| A      | \[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\] = \[
\begin{pmatrix}
0
\end{pmatrix}
\] | blue |
| B      | \[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\] = \[
\begin{pmatrix}
1
\end{pmatrix}
\] | blue | Lose |
| C      | \[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\] = \[
\begin{pmatrix}
1
\end{pmatrix}
\] | blue |

| A      | \[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\] = \[
\begin{pmatrix}
1
\end{pmatrix}
\] | pass |
| B      | \[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\] = \[
\begin{pmatrix}
0
\end{pmatrix}
\] | pass | Win! |
| C      | \[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\] = \[
\begin{pmatrix}
1
\end{pmatrix}
\] | blue |
\[
\begin{align*}
A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \text{pass} \\
\{ \text{Trial #3:} \} \\
B \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{blue} \\
C \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{pass} \\
\end{align*}
\]

\[
\begin{align*}
A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \text{red} \\
\{ \text{Trial #4:} \} \\
B \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{pass} \\
C \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{pass} \\
\end{align*}
\]

\[
\begin{align*}
A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{blue} \\
\{ \text{Trial #5:} \} \\
B \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \text{pass} \\
C \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{pass} \\
\end{align*}
\]
\[
\begin{align*}
\text{Trial #6:} & \quad \text{B}
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix} \\
\text{red} & \quad \{ \text{Win!} \}
\end{align*}
\]

\[
\begin{align*}
\text{Trial #7:} & \quad \text{B}
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix} \\
\text{pass} & \quad \{ \text{Win!} \}
\end{align*}
\]

\[
\begin{align*}
\text{Trial #8:} & \quad \text{B}
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix} \\
\text{red} & \quad \{ \text{Lose} \}
\end{align*}
\]